Automatica 48 (2012) 2882–2887

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# Understanding the impact of churn in dynamic oligopoly markets☆

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ARTICLE INFO

*Article history:*
Received 19 August 2011
Received in revised form
30 January 2012
Accepted 1 May 2012
Available online 18 September 2012

*Keywords:*
Advertising
Churn
Oligopoly
Differential games
Optimization

ABSTRACT

We incorporate the effects of churn, which refers to customers switching to competing brands, in a dynamic model of advertising for oligopoly markets. Each firm's market share depends not only on its own and competitors' advertising decisions, but also on market churn. Applying differential game theory, we derive a feedback Nash equilibrium under symmetric and asymmetric competition. We obtain explicit solutions and discover the counter-intuitive result that, as market churn increases, firms should decrease advertising rather than increase it to counteract the impact of churn.



## 1. Introduction

Customer churn refers to the attrition or turnover of customers; it is antonymous with customer retention or customer loyalty. All firms across diverse industries experience customer churn (Neslin, Gupta, Kamakura, Lu, & Mason, 2006). To replace churning customers, firms incur substantial costs of advertising and other marketing efforts in acquiring new customers. Recognizing its importance, empirical studies (e.g. KhakAbi, Gholamian, & Namvar, 2010) focus on quantifying the magnitude of churn, but the resulting insights are data dependent. The extant literature lacks theoretical studies that offer general insights into the impact of churn. For example, should brands increase or decrease advertising as churn increases in dynamic oligopoly markets? Although managers may increase advertising expenditures to counteract the negative influence of churn, they need to consider and balance the positive influence of churn as customers from competing brands switch to their brands. Consequently, a priori, the impact of churn on advertising expenditures and, from that, on profitability, is ambiguous and thus requires systematic analysis.

In many industries, advertising plays an important role when competing for market shares over time. Some examples are provided by the markets for cola drinks, beers, and cigarettes (see Erickson, 1992; Fruchter & Kalish, 1997). Each firm advertises to increase market share, while the competitor's advertising reduces it. To capture the carryover dynamics of advertising and competitive interactions, these studies apply differential game theory in order to understand the best course of action for each firm while taking into consideration the response of its competitors.

This analysis of dynamic advertising competition has been extended beyond two firms. Teng and Thompson (1983) and Dockner and Jørgensen (1992) develop oligopoly models where sales grow over time via innovation diffusion dynamics, the market reaches saturation after some time, and competitive advertising affects market saturation to influence sales indirectly. They show that advertising should decrease over time due to the saturation effect. Using goodwill accumulation dynamics, Fershtman (1984) finds that firms should decrease advertising, except possibly for the market share leader, as the number of firms increases. Erickson (1995, 2003) uses the method of dynamic conjectural variations to study oligopoly markets. For the case of three symmetric competitors and zero discount rate, Erickson (2003, p.103) obtains a simple solution for the advertising rate, showing that it increases with the profit margin and the conjectured lack of response of the competitor; and it decreases with the concavity of the market share response to advertising. Fruchter (1999) extends the closed-loop duopoly analysis of Fruchter and Kalish (1997) to an oligopoly, and demonstrates that treating an oligopoly as a two-player game by aggregating all the rival firms results in suboptimal advertising.

☆ The material in this paper was presented at the 13th IFAC Symposium on Information Control Problems in Manufacturing (INCOM '2009), June 3–5, 2009, Moscow, Russian Federation. This paper was recommended for publication in revised form by Associate Editor Michèle Breton under the direction of Editor Berç Rüstem.

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doi:10.1016/j.automatica.2012.08.031

**Table 1**
List of variables and parameters.

| Notation | Explanation |
|---|---|
| $x_i(t) \in [0, 1]$ | Market share of firm $i$, $i \in \mathbf{I} \equiv \{1, 2, \ldots, n\}$, at time $t$. |
| $u_i(\mathbf{x(t)}) \geq 0$ | Advertising rate of firm $i$ at time $t$, where $\mathbf{x(t)} \equiv (x_1(t), x_2(t), \ldots, x_n(t))'$. |
| $\rho_i > 0$ | Advertising effectiveness parameter. |
| $\xi_i > 0$ | Competitive advertising decay parameter. |
| $\delta > 0$ | Churn rate parameter. |
| $\mu_i \in (0, 1)$ | Churn point, where $\sum_{i\in\mathbf{I}} \mu_i = 1$. |
| $r_i > 0$ | Discount rate for firm $i$. |
| $C(u_i)$ | Cost of advertising control $u_i$, parametrized as $c_i u_i^2$, $c_i > 0$. |
| $m_i > 0$ | Industry sales multiplied by the per unit profit margin for firm $i$. |
| $V_i$ | Value function for firm $i$. |
| $\alpha, \beta, \gamma, \phi$ | Components of the value function. |
| $B_i$ | A useful intermediate term; $B_i \equiv \rho_i^2 \left(n\phi_i^i - \sum_{k\in\mathbf{I}} \phi_k^i\right)/2c_i(n-1)^2 + \delta/n$. |

This paper extends a monopoly model given by Sethi (1983) to the oligopoly context, providing explicit closed-loop solutions. Naik, Prasad, and Sethi (2008) likewise extend the Sethi model to describe awareness growth for car brands, but they ignore the effects of market churn, which is the focus of this study. We solve an infinite horizon differential game to derive the feedback Nash equilibrium and gain insights into the effects of churn on advertising expenditure, market share, and profitability.

## 2. The model

We consider an $n$-firm oligopoly in a mature product category (telephone companies, Internet service providers, banks, insurance firms, etc.) such that the total sales of the category are relatively stable and a customer leaving one firm joins another. Let $x_i(t)$ denote the market share of firm $i$, $i \in \mathbf{I} \equiv \{1, 2, \ldots, n\}$, at time $t$ and $n \geq 2$. We will use the notation listed in Table 1.

Firm $i$'s objective is to maximize its long-run discounted profit,

$$\underset{u_i \geq 0}{\text{Max}} \left\{ V_i(x_{i0}) = \int_0^\infty e^{-r_i t}[m_i x_i(t) - c_i u_i(t)^2]dt \right\}, \quad (1)$$

subject to market share dynamics given by

$$\frac{dx_i}{dt} = \rho_i u_i \sqrt{1-x_i} - \sum_{j\in\mathbf{I}, j\neq i} \xi_j u_j \sqrt{1-x_j} - \delta(x_i - \mu_i), \quad \forall i \in \mathbf{I}. \quad (2)$$

The first term on the right hand side is the market share gain due to advertising; the second term is the market share loss due to competitive advertising; and the last term is market share churn. Advertising influences non-adopters to purchase the advertised brand, and churn acts on adopters as in the duopoly model of Prasad and Sethi (2004). Churn exerts a pull on market shares towards a specified point $(\mu_1, \mu_2, \ldots \mu_n)$. When $\mu_i = 1/n\ \forall i$, churn equalizes the market shares; when $\mu_i$ differs across firms, it favours some firms over others.

Eq. (2) extends the Sethi (1983) model of a monopolistic firm, $\dot{x}(t) = \rho u(t)\sqrt{1-x(t)} - \delta x(t)$, where the decay term $\delta$, also present in Vidale and Wolfe (1957) and other models, captures effects such as product obsolescence and forgetting. Consistent with how decay is modelled in the literature, churn is proportional to market share. When the churn term is omitted and a duopoly considered, the dynamics is identical to Sorger (1989), which has been empirically validated by Chintagunta and Jain (1995).

The square root over the untapped market potential makes the analysis tractable by making the value function linear. Sorger (1989) interprets the interaction term $x(1-x)$ in the approximation $\sqrt{1-x} \approx 1 - x + x(1-x)$ as the word of mouth effect due to current customers ($x$) influencing customers in the untapped market $(1-x)$.

Market shares should (i) be non-negative and (ii) sum to unity at each instant. We furnish a sufficient condition in Section 3 for these two requirements to be satisfied. Because market shares sum to unity, i.e., $\sum_{i\in\mathbf{I}} x_i = 1$, we observe that $\sum_{i\in\mathbf{I}} dx_i/dt = 0$, which implies a restriction on the parameters: $\xi_i = \rho_i/(n-1)$. Then, Eq. (2) can be rewritten as

$$\frac{dx_i}{dt} = \frac{n}{n-1}\rho_i u_i \sqrt{1-x_i} - \frac{1}{n-1}\sum_{j\in\mathbf{I}} \rho_j u_j \sqrt{1-x_j} - \delta(x_i - \mu_i). \quad (3)$$

## 3. Analysis

We perform the analysis by solving the Hamilton–Jacobi–Bellman (HJB) equation for each firm. Note that the optimal advertising control $u_i$ will be non-negative for all firms because the cost term $c_i u_i^2$ is always positive.

The nature of the dynamics ensures that the market share of each firm never exceeds 1. However, we need to ensure that the market share of each firm should remain non-negative. We will derive the situations for market shares to remain non-negative given optimal advertising decisions in a later section. This approach is preferable to constraining the market shares *ex ante* since the constrained maximization problem is analytically intractable. However, as discussed in Naik et al. (2008), this approach may yield negative value functions when the number of firms exceeds an upper bound which is at least 3 in the symmetric case.

In Theorem 1, we provide the Nash equilibrium solution of the advertising game.

**Theorem 1.** *Let the $\varphi$'s be determined from the relations*

$$r_i\left(\sum_{j=0}^{n}\varphi_j^i\right) = m_i - \delta\sum_{j\in\mathbf{I}}\varphi_j^i(1-\mu_j),$$

$$(r_i+\delta)\varphi_i^i = m_i - \frac{\rho_i^2}{4c_i(n-1)^2}\left(n\varphi_i^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right)^2,$$

$$(r_i+\delta)\varphi_j^i = -\frac{\rho_j^2}{2c_j(n-1)^2}\left(n\varphi_j^j - \sum_{k\in\mathbf{I}}\varphi_k^j\right)\left(n\varphi_j^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right), \quad \forall j \in \mathbf{I}, j \neq i.$$

*Then, for the differential game given by* (1) *and* (3), *the optimal feedback advertising rate for firm $i$ is*

$$u_i* = \frac{\rho_i\sqrt{1-x_i}}{2c_i(n-1)}\left(n\varphi_i^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right),$$

and the value function is

$$V_i = \varphi_0^i + \sum_{j=1}^{n} \varphi_j^i x_j,$$

provided this solution results in non-negative market shares and value functions. (All proofs are given in Appendix.)

Remarkably, the HJB equations could be solved analytically to yield a relatively simple optimal control, as a consequence of the linear value function solution. Thus, we gain the insight that the optimal advertising expenditures are inversely proportional to market shares. In other words, a smaller firm should spend more aggressively than a larger firm. In proceeding, we assume that $\mu_i = 1/n\ \forall i$.

### 3.1. An illustration of Theorem 1

To illustrate the application of Theorem 1, we examine a triopoly. To simplify the exposition, we restrict the asymmetry to one firm, keeping the remaining two firms symmetric. Thus, $\mathbf{I} \equiv \{1, 2, 3\}$, and let firm 1 be the asymmetric firm. Let $m_2 = m_3 = m, r_2 = r_3 = r, c_2 = c_3 = c$. We simplify the notation by dropping the superscript notation and proceeding with the following notation for the linear value functions:

$$\begin{cases} V_1 = \alpha_1 + \beta_1 x_1 + \gamma_1 x_2 + \gamma_1 x_3, \\ V_2 = \alpha + \gamma x_1 + \beta x_2 + \eta x_3, \\ V_3 = \alpha + \gamma x_1 + \eta x_2 + \beta x_3. \end{cases}$$

We then apply Theorem 1. The advertising decisions are $u_1{*} = \frac{\rho_1\sqrt{1-x_1}}{2c_1}(\beta_1 - \gamma_1)$, and $u_i{*} = \frac{\rho\sqrt{1-x_i}}{4c}(2\beta - \gamma - \eta)$, $i = \{2, 3\}$. For the seven unknown parameters $\alpha_1, \beta_1, \gamma_1, \alpha, \beta, \gamma$ and $\eta$, seven equations are obtained:

$$\begin{cases} r_1(\alpha_1 + \beta_1 + 2\gamma_1) = m_1 - \dfrac{2\delta}{3}(\beta_1 + 2\gamma_1), \\ r(\alpha + \beta + \gamma + \eta) = m - \dfrac{2\delta}{3}(\beta + \gamma + \eta), \\ (r_1 + \delta)\beta_1 = m_1 - \dfrac{\rho_1^2}{4c_1}(\beta_1 - \gamma_1)^2, \\ (r + \delta)\beta = m - \dfrac{\rho^2}{16c}(2\beta - \gamma - \eta)^2, \\ (r_1 + \delta)\gamma_1 = -\dfrac{\rho^2}{8c}(2\beta - \gamma - \eta)(\gamma_1 - \beta_1), \\ (r + \delta)\gamma = -\dfrac{\rho_1^2}{4c_1}(\beta_1 - \gamma_1)(2\gamma - \beta - \eta), \\ (r + \delta)\eta = -\dfrac{\rho^2}{8c}(2\beta - \gamma - \eta)(2\eta - \beta - \gamma). \end{cases}$$

For illustration, let $m_i = 1.1, m = 1, r_i = 0.05, \forall i$, and $\rho_i^2/c_i = 1, \forall i$. We solve the simultaneous equations using Maple for different values of the churn parameter.

The unique solution is identified by eliminating solutions that have imaginary values or yield negative value functions. On the basis of this, Fig. 1 shows a decline in advertising and increase in the value function with the churn parameter.

### 3.2. Steady-state market shares

Substituting the optimal advertising control expression from Theorem 1 into the original dynamics, we get $\forall i \in \mathbf{I}$,

$$\frac{dx_i}{dt} = \frac{n\rho_i^2(1 - x_i)}{2c_i(n-1)^2}\left(n\phi_i^i - \sum_{k\in\mathbf{I}}\phi_k^i\right) - \sum_{j\in\mathbf{I}}\frac{\rho_j^2(1 - x_j)}{2c_j(n-1)^2}\left(n\phi_j^j - \sum_{k\in\mathbf{I}}\phi_k^j\right) - \delta\left(x_i - \frac{1}{n}\right).$$

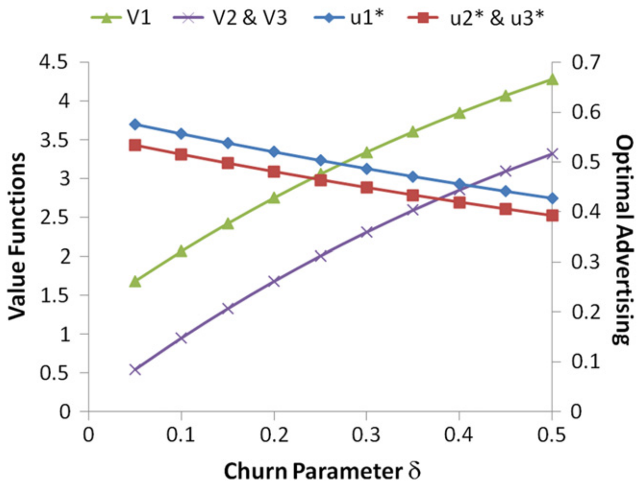


**Fig. 1.** Value functions and advertising rates versus $\delta$.

In the steady state, $(dx_i/dt) = 0$, and we can solve for the steady-state market shares.

**Theorem 2.** *Let us define*

$$B_i \equiv \frac{\rho_i^2}{2c_i(n-1)^2}\left(n\phi_i^i - \sum_{k\in\mathbf{I}}\phi_k^i\right) + \frac{\delta}{n};$$

*then the following results are obtained:*

(a) *The market share dynamics can be expressed as*

$$\frac{dx_i}{dt} = n(1 - x_i)B_i - \sum_{j\in\mathbf{I}}(1 - x_j)B_j, \quad \forall i \in \mathbf{I}.$$

(b) *The unique solution for the market share vector is* $\mathbf{x(t)} = (\mathbf{I} - e^{-t\mathbf{B}})\mathbf{1} + e^{t\mathbf{B}}\mathbf{x(0)}$, *where*

$$\mathbf{B} = \begin{pmatrix} -(n-1)B_1 & B_2 & \cdots & B_n \\ B_1 & -(n-1)B_2 & \cdots & B_n \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \cdots & -(n-1)B_n \end{pmatrix}.$$

(c) *The steady-state market share of the ith firm is given by the formula*

$$\bar{x}_i = 1 - \frac{n-1}{B_i \sum_{j\in\mathbf{I}} \frac{1}{B_j}}, \quad \forall i \in \mathbf{I}.$$

Thus, for a triopoly, the steady-state market shares are

$$\bar{x}_i = 1 - \frac{2B_1B_2B_3}{B_i(B_1B_2 + B_2B_3 + B_1B_3)}, \quad i = 1, 2, 3.$$

Note that if the $B_i$'s are equal for all firms, then each firm gets an equal share of the market, which is decreasing in the number of firms in the industry.

### 3.3. An illustration of Theorem 2

We continue the example in 3.1 and use Theorem 2(a) and (b) to obtain market share trajectories and the steady state using numerical simulations. To obtain the trajectory, one may apply numerical methods of solving the system of linear differential equations in Theorem 2(a), such as the classical Runge–Kutta method. However, since we have a solution to the system of equations, we show how it may be directly applied when the $B_i$

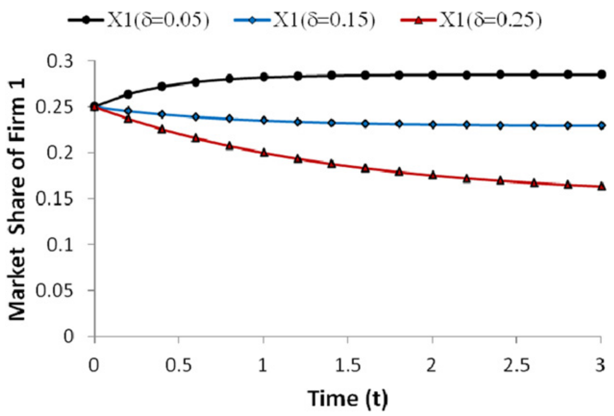


**Fig. 2.** Market share versus churn.

parameters for the industry are known, which may be estimated via historical data or expert judgment.

Continuing the illustration in Section 3.1, if $\delta = 0.05$ then $B_1 = 0.794$, $B_2 = B_3 = 0.884$ and let $\mathbf{x(0)} = (0.25, 0.5, 0.25)$. The market shares are given by $\mathbf{y(t)} = e^{t\mathbf{B}}\mathbf{y(0)}$ where $\mathbf{y(t)} = \mathbf{1} - \mathbf{x(t)}$. It is convenient to diagonalize the matrix

$$\mathbf{B} = \begin{pmatrix} -1.59 & 0.884 & 0.884 \\ 0.794 & -1.768 & 0.884 \\ 0.794 & 0.884 & -1.768 \end{pmatrix},$$

as shown (to two significant digits) below:

$$\mathbf{B} = \mathbf{PAP^{-1}} = \begin{pmatrix} -0.89 & -0.80 & 0.00 \\ 0.45 & -0.72 & 0.50 \\ 0.45 & -0.72 & 0.50 \end{pmatrix} \times \begin{pmatrix} -2.47 & 0 & 0 \\ 0 & 0.00 & 0 \\ 0 & 0 & -2.65 \end{pmatrix} \begin{pmatrix} -0.72 & 0.40 & 0.40 \\ -0.45 & -0.45 & -0.45 \\ 0.00 & -1.00 & 1.00 \end{pmatrix}.$$

From matrix theory, $\mathbf{y(t)} = \mathbf{P}e^{t\mathbf{A}}\mathbf{P^{-1}y(0)}$. As $\mathbf{x(t)} = \mathbf{1} - \mathbf{y(t)}$, we get

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \approx \begin{pmatrix} 0.285 - 0.035e^{-2.471t} \\ 0.358 + 0.017e^{-2.471t} + 0.125e^{-2.652t} \\ 0.358 + 0.017e^{-2.471t} - 0.125e^{-2.652t} \end{pmatrix}.$$

This, of course, agrees with Theorem 2(c) on the steady-state market shares, i.e., $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0.285, 0.358, 0.358)$.

We repeat this exercise for $\delta = 0.15$ and $\delta = 0.25$ and plot the market shares for firm 1 over time in Fig. 2.

While Fig. 1 suggested that churn benefits profit due to reduced advertising, Fig. 2 shows that its effect on the market share of firm 1 is negative, and thus that on the other firms is positive, and it is ambiguous overall.

### 3.4. Examination of non-negativity requirements

We now consider the requirement that the market shares are non-negative and derive the sufficient condition. On the basis of the previous section, the steady-state market shares trajectories are given by

$$dx_i/dt = n(1 - x_i)B_i - \sum_{j\in\mathbf{I}}(1 - x_j)B_j, \quad \forall i \in \mathrm{I}. \tag{4}$$

For the market share of firm $i$ to be less than 1, the required condition is that $dx_i/dt \le 0$ when $x_i = 1$. An examination of (4) shows that this condition is always satisfied.

For the market share to be greater than 0, the required condition is that $dx_i/dt \ge 0$ when $x_i = 0$. This condition is always satisfied if $n = 2$ since $x_j = 1$ and $dx_i/dt = nB_i$. Let us, however, consider the case where $n > 2$. For this analysis, let us renumber the firms such that $B_1 \le B_2 \le \cdots \le B_n$. We have the following result:

**Result 1.** The requirement

$$B_1 \ge \frac{1}{(n-1)} \sum_{j=3}^{n} B_j$$

is a sufficient condition for market shares to be non-negative.

Thus, in the three-firm and four-firm cases, $B_1 \ge B_3/2$ and $B_1 \ge (B_3 + B_4)/3$ are the required conditions, respectively. Let us now consider the extreme case of market structure where $B_3 = B_4 = \cdots = B_n = B$. Then the condition is $B_1 \ge \frac{(n-2)B}{n-1}$. Since $\lim_{n\to\infty}(n-2)/(n-1) = 1$, we get $B_1 \ge B$. But by definition $B_1 \le B$; thus $B_1 = B$ is the required condition in the limit as the number of firm becomes very large. This allows a progressively decreasing spread between the highest and lowest valuation firms as the number of firms increases. Hence, we cannot study extremely asymmetric markets. However, with symmetric firms, the requirement is always satisfied.

### 3.5. Symmetric firms

We will solve for the unknown parameters in the case where firms are symmetric, i.e., $\forall i \in \mathbf{I}, m_i = m, \rho_i = \rho, r_i = r$, and, thus, $\phi_0^i = \alpha, \phi_i^i = \beta, \phi_j^i(\forall j \ne i) = \gamma$. By construction, a unique, explicit solution to the differential game exists. By symmetry, $B_i = B, \forall i$. Hence, the dynamics simplifies to $dx_i/dt = nB((1/n) - x_i), \forall i \in \mathbf{I}$, revealing that each firm moves monotonically from its initial market share to the steady state market share of $1/n$. Further results are presented below.

**Corollary 1.** *For n symmetric firms:*

(a) *The value function is* $V_i = (\alpha + \gamma) + (\beta - \gamma)x_i$, *where*

$$\beta - \gamma = \frac{2m}{(r+\delta) + \sqrt{(r+\delta)^2 + \frac{m\rho^2(n+1)}{c(n-1)}}} > 0.$$

(b) *The optimal advertising rate is* $u_i^* = \frac{\rho\sqrt{1-x_i}}{2c}(\beta - \gamma)$.

(c) *Parameters* $\alpha, \beta, \gamma$ *are determined from the relations*

$$\alpha = \frac{m}{r} - \frac{r + \delta - \delta/n}{r(n+1)}\left(\frac{2mn}{r+\delta} - (n-1)(\beta - \gamma)\right),$$

$$\beta = \frac{1}{n+1}\left(\frac{2m}{r+\delta} + (n-1)(\beta - \gamma)\right),$$

$$\gamma = \frac{2}{n+1}\left(\frac{m}{r+\delta} - (\beta - \gamma)\right).$$

Noting that $\frac{\partial(\beta-\gamma)}{\partial\delta} < 0$, the following results follow from parts (b) and (c) of this corollary:

$$\frac{\partial u*}{\partial\delta} < 0, \qquad \frac{\partial u*}{\partial x} < 0, \qquad \frac{\partial^2 u*}{\partial x \partial\delta} > 0. \tag{5}$$

From $\partial u*/\partial\delta < 0$ we gain a novel substantive insight into the impact of market churn on advertising expenditures. Specifically, firms should decrease advertising as market churn increases rather than increasing it to counteract the negative influence of churn. Why? Because managers should account for the positive influence of "churn-in" as customers from the competing brands switch to their brands.

While earlier literature has found the inverse effect between market share and advertising (i.e., $\partial u*/\partial x < 0$), we find that it holds even in the presence of churn but is mitigated by it, as evidenced by the cross partial derivative ($u^*_{x\delta} > 0$).

Finally, the sensitivity of the value function to the churn parameter is hard to untangle, but if we consider symmetric initial conditions, then

$$V(1/n) = \int_0^\infty e^{-rt}[m/n - cu^{*2}]dt = \frac{m}{rn} - \frac{\rho^2(n-1)}{4crn}(\beta-\gamma)^2. \quad (6)$$

This function has a positive slope with respect to $\delta$ as a consequence of churn reducing the optimal advertising. To see that $V(1/n)$ is positive, we substitute $\beta - \gamma$ and simplify to get

$$n(3-n) + \frac{2\sqrt{c}(r+\delta)(n-1)^{3/2}}{\rho\sqrt{m}} \geq 0, \quad (7)$$

which is satisfied for $n \leq 3$ at least.

## 4. Conclusions

Customer turnover rates (i.e., churn) vary from the single digits to well over 50% for some telecom services industries. Attrition and acquisition of customers through advertising imposes a significant cost on firms. American companies alone spend over \$250 billion annually and often suboptimally. Our analysis of optimal advertising expenditures in an oligopoly market in the presence of market churn provides a counter-intuitive result to help companies reduce advertising to combat market churn.

We obtain the feedback Nash equilibrium advertising strategies and show that advertising is inversely proportional to market share. For symmetric oligopoly markets, we present an explicit closed-loop solution for the advertising expenditures of all firms. The optimal advertising decreases as the churn rate increases. The steady-state market shares are given by a simple expression based on the strengths of the different brands.

It would be worthwhile to explore various causes of churn, such as pricing and promotions. Also we obtained analytical results for only symmetric firms. Although our numerical results for asymmetric firms agree with those analytical results, a conclusive proof of the sensitivity of advertising and profits to the churn parameter is unavailable and merits further exploration.

## Appendix

**Proof of Theorem 1.** The Hamilton–Jacobi–Bellman (HJB) equation for firm $i$ is given by

$$r_iV_i = \max_{u_i} m_ix_i - c_iu_i^2 + \sum_{j\in\mathbf{I}} \frac{\rho_j u_j\sqrt{1-x_j}}{n-1}\left(n\frac{\partial V_i}{\partial x_j} - \sum_{k\in\mathbf{I}}\frac{\partial V_i}{\partial x_k}\right) - \sum_{j\in\mathbf{I}}\frac{\partial V_i}{\partial x_j}\delta(x_j-\mu_j).$$

We obtain the optimal feedback controls

$$u_i* = \max\left\{0, \frac{\rho_i\sqrt{1-x_i}}{2c_i(n-1)}\left(n\frac{\partial V_i}{\partial x_i} - \sum_{k\in\mathbf{I}}\frac{\partial V_i}{\partial x_k}\right)\right\}.$$

Anticipating that the controls will be shown to be positive, we insert these controls into the HJB equations to obtain, for firm $i$,

$$r_iV_i = m_ix_i - \frac{\rho_i^2(1-x_i)}{4c_i(n-1)^2}\left(n\frac{\partial V_i}{\partial x_i} - \sum_{k\in\mathbf{I}}\frac{\partial V_i}{\partial x_k}\right)^2 - \sum_{j\in\mathbf{I}}\frac{\partial V_i}{\partial x_j}\delta(x_j-\mu_j) + \sum_{j\in\mathbf{I}}\frac{\rho_j^2(1-x_j)}{2c_j(n-1)^2} \times \left(n\frac{\partial V_j}{\partial x_j} - \sum_{k\in\mathbf{I}}\frac{\partial V_j}{\partial x_k}\right)\left(n\frac{\partial V_i}{\partial x_j} - \sum_{k\in\mathbf{I}}\frac{\partial V_i}{\partial x_k}\right).$$

To solve these $n$ simultaneous partial differential equations, we use the following linear value functions:

$$V_i = \phi_0^i + \sum_{j=1}^n \phi_j^i x_j, \quad \forall i \in \mathbf{I}$$

observing that they satisfy the Hamilton–Jacobi equations. Thus, there are a total of $n+1$ unknown parameters for each of the $n$ firms. To determine these, we insert the value function into the Hamilton–Jacobi equation and obtain, $\forall i \in \mathbf{I}$,

$$r_i\left(\varphi_0^i + \sum_{j=1}^n \varphi_j^i x_j\right) = m_ix_i - \frac{\rho_i^2(1-x_i)}{4c_i(n-1)^2}\left(n\varphi_i^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right)^2 + \sum_{j\in\mathbf{I}}\frac{\rho_j^2(1-x_j)}{2c_j(n-1)^2}\left(n\varphi_j^j - \sum_{k\in\mathbf{I}}\varphi_k^j\right)\left(n\varphi_j^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right) - \sum_{j\in\mathbf{I}}\delta\varphi_j^i x_j + \delta\sum_{j\in\mathbf{I}}\varphi_j^i\mu_j$$

$$\Rightarrow r_i\left(\sum_{j=0}^n \varphi_j^i - \sum_{j=1}^n \varphi_j^i(1-x_j)\right) = m_i - m_i(1-x_i) - \frac{\rho_i^2(1-x_i)}{4c_i(n-1)^2}\left(n\varphi_i^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right)^2 + \sum_{j\in\mathbf{I}}\frac{\rho_j^2(1-x_j)}{2c_j(n-1)^2}\left(n\varphi_j^j - \sum_{k\in\mathbf{I}}\varphi_k^j\right)\left(n\varphi_j^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right) + \delta\sum_{j\in\mathbf{I}}\varphi_j^i(1-x_j) - \delta\sum_{j\in\mathbf{I}}\varphi_j^i(1-\mu_j).$$

Equating powers of $(1-x_i)$, we get the $n(n+1)$ required equations that determine the $n(n+1)$ unknown coefficients. □

**Proof of Theorem 2.** Part (a): We can write

$$\frac{dx_i}{dt} = n(1-x_i)A_i - \sum_{j\in\mathbf{I}}(1-x_j)A_j - \delta(x_i - 1/n),$$

where we define

$$A_i \equiv \frac{\rho_i^2}{2c_i(n-1)^2}\left(n\varphi_i^i - \sum_{k\in\mathbf{I}}\varphi_k^i\right).$$

This can be further simplified to

$$\frac{dx_i}{dt} = n(1-x_i)B_i - \sum_{j\in\mathbf{I}}(1-x_j)B_j$$

where

$$B_i \equiv A_i + \frac{\delta}{n} = \frac{\rho_i^2}{2c_i(n-1)^2}\left(n\phi_i^i - \sum_{k\in\mathbf{I}}\phi_k^i\right) + \frac{\delta}{n}.$$

Part (b): The dynamics may further be written as $\frac{d\mathbf{y(t)}}{dt} = \mathbf{B}y(t)$ where $y_i \equiv 1 - x_i$. The unique solution of this system of equations is $\mathbf{y(t)} = e^{t\mathbf{B}}\mathbf{y}(0)$, or $\mathbf{x(t)} = (\mathbf{I} - e^{-t\mathbf{B}})\mathbf{1} + e^{t\mathbf{B}}\mathbf{x(0)}$, where $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is a column vector of 1's, and $e^{t\mathbf{B}} = \mathbf{I} + t\mathbf{B} + \frac{t^2\mathbf{B}^2}{2!} + \cdots = \sum_{k=0}^\infty \frac{(t\mathbf{B})^k}{k!}$.

Part (c): In the steady state, $(dx_i/dt) = 0$. Let $\bar{x}_i$ denote the market share in the steady state for firm $i$. From part (a),

$$1-\bar{x}_i = \frac{1}{nB_i}\sum_{j\in\mathbf{I}}(1-\bar{x}_j)B_j$$

$$\Rightarrow \sum_{i\in\mathbf{I}}(1-\bar{x}_i) = \sum_{i\in\mathbf{I}}\frac{1}{nB_i}\sum_{j\in\mathbf{I}}(1-\bar{x}_j)B_j$$

$$\Rightarrow \frac{n-1}{B_i\sum_{i\in\mathbf{I}}\frac{1}{B_i}} = \frac{1}{nB_i}\sum_{j\in\mathbf{I}}(1-\bar{x}_j)B_j.$$

Inserting back the last expression gives the desired result. □

**Proof of Result 1.** We require that $(dx_i/dt) \geq 0$ when $x_i = 0$. Inserting this into (4), the required condition may be restated as

$$nB_i \geq \sum_{j\in\mathbf{I}}(1-x_j)B_j, \quad \forall i \in \mathbf{I}.$$

To obtain a sufficient condition, we will consider the most pessimistic condition. Renumber the firms such that $B_1 \leq B_2 \leq \cdots \leq B_n$. Then, we pick the smallest value of the left hand side and the largest value of the right hand side. This gives the required condition

$$nB_1 \geq \sum_{j\in\mathbf{I}}B_j - B_2, \quad \text{or} \quad (n-1)B_1 \geq \sum_{j=3}^{n}B_j. \quad \square$$

**Proof of Corollary 1.** From Theorem 1, we have the relationships

$$r\alpha + \left(r+\delta-\frac{\delta}{n}\right)(\beta+(n-1)\gamma) = m,$$

$$(r+\delta)\beta = m - \frac{\rho^2(\beta-\gamma)^2}{4c},$$

$$(r+\delta)\gamma = \frac{\rho^2(\beta-\gamma)^2}{2c(n-1)}.$$

Subtracting the third from the second equation, one obtains a quadratic equation in $\beta-\gamma$, which yields

$$\beta-\gamma = \left(\pm\sqrt{(r+\delta)^2+\frac{m\rho^2(n+1)}{c(n-1)}}-(r+\delta)\right)\Bigg/\left(\frac{\rho^2(n+1)}{2c(n-1)}\right).$$

We express the value function as $V_i = (\alpha+\gamma)+(\beta-\gamma)x_i$. We expect $\beta-\gamma > 0$, and take the positive root. This yields

$$\beta-\gamma = \frac{2m}{\sqrt{(r+\delta)^2+\frac{m\rho^2(n+1)}{c(n-1)}}+(r+\delta)}.$$

We express the unknowns in terms of $\beta-\gamma$ and substitute into the first equation to get

$$\alpha = \frac{m}{r} - \frac{r+\delta-\delta/n}{r(n+1)}\left(\frac{2mn}{r+\delta}-(n-1)(\beta-\gamma)\right). \quad \square$$

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