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This article develops a method for optimal allocation of resources based on an empirically validated model of how national and regional advertising generate sales over time. The authors derive the profit-maximizing total budget, its optimal split between national and regional spends, and its optimal allocation across multiple regions. They formulate a spatiotemporal model that accounts for spatial and serial dependence, spatial heterogeneity, neighborhood effects, and sales dynamics. Because of spatial and serial dependence, correlated multivariate Brownian motion drives the sales dynamics, resulting in a second-order differential equation for the Hamilton–Jacobi–Bellman (HJB) equation with multiple states (i.e., regional sales) and multiple controls (i.e., regional and national advertising expenditures). By solving the HJB equation analytically, the authors furnish closed-form expressions for the optimal total budget and its regional allocations. In addition, they develop a method to estimate the proposed model and apply it to market data from a leading German cosmetics company. Using the estimated parameters, they evaluate the optimal budget and allocations. Comparing them with actual company policy, the proposed approach enhances profit by 5.07%, and it not only identifies which regions under- or overspend but also reveals how much budget to shift from national to regional advertising (or vice versa).

Keywords: advertising, budget allocation, spatiotemporal model, neighborhood effects, spatial dependence, spatial heterogeneity

Spatiotemporal Allocation of Advertising Budgets

In our recent meeting with the chief marketing officer of a leading cosmetics firm, she broached the topic of how to spend €100 million to advertise a brand, whether 100 million is the "right" sum, how much of it should be set aside for national advertising, and how to allocate it across the
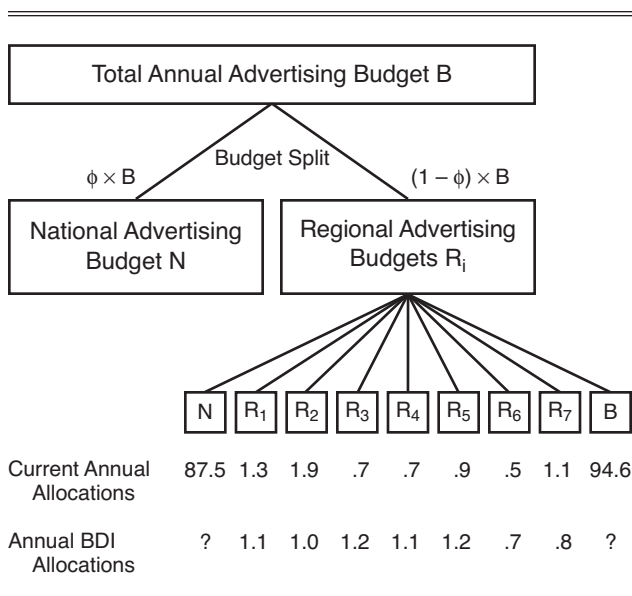
seven Nielsen regions of Germany. When we asked what the firm does now, she revealed (see Figure 1) the actual allocation as well as the spending plan based on marketing textbooks, which relies on a brand development index (BDI) as the basis for allocation, though she noted that this BDI plan recommends neither the total sum nor how much to spend on national advertisements, let alone whether it is optimal.

The BDI-based approach results in advertising spend proportional to the per capita sales in each region. To assess the optimality of these allocation decisions, we require both the response model and profit function, which the BDI-based approach lacks, a point to which Lodish (2007, p. 24) alludes. This drawback highlights the need for a method for optimal allocation of resources based on (1) an empirically validated model of how national and regional advertising generates sales over time and (2) a normative analysis that derives the profit-maximizing total budget, its optimal split between national and regional spends, and its optimal allo-

Figure 1
SPATIAL BUDGETING AND ALLOCATION SETTING

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                    ┌──────────────────────────────────┐
                    │  Total Annual Advertising Budget B │
                    └──────────────────────────────────┘
                              Budget Split
              φ × B                          (1 − φ) × B
        ┌──────────────────┐        ┌──────────────────┐
        │ National Advertising │     │ Regional Advertising │
        │    Budget N       │        │   Budgets Rᵢ       │
        └──────────────────┘        └──────────────────┘
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	N	R_1	R_2	R_3	R_4	R_5	R_6	R_7	B
Current Annual Allocations	87.5	1.3	1.9	.7	.7	.9	.5	1.1	94.6
Annual BDI Allocations	?	1.1	1.0	1.2	1.1	1.2	.7	.8	?

Notes: All amounts are in millions of euros.

cation across the multiple regions. This article develops a method to answer these questions.

Previous research has built spatial models to capture variations in brand performance across regions (e.g., Ataman, Mela, and Van Heerde 2007; Bell, Ho, and Tang 1998; Bronnenberg, Dhar, and Dubé 2007a, b; Bronnenberg and Mahajan 2001; Chan, Padmanabhan, and Seetharaman 2007; Thomadsen 2007). These models account for spatial heterogeneity (i.e., marketing response differs across regions), neighborhood effects (i.e., past sales in neighboring regions affect the focal region), and spatial dependency (i.e., errors are correlated across regions) but ignore the dynamic effects of advertising. To account for dynamics, spatiotemporal models have recently been introduced in marketing (e.g., Albuquerque, Bronnenberg, and Corbett 2007; Bell and Song 2007; Choi, Hui, and Bell 2010); however, they do not provide closed-form budgeting or allocation expressions. In contrast, several studies (e.g., Doyle and Saunders 1990; Naik and Raman 2003; Skiera and Albers 1998) that provide normative findings disregard the spatial effects, which, when ignored, lead to inaccurate forecasts (Giacomini and Granger 2004). Thus, as the literature review shows, no study estimates a spatiotemporal model of advertising and derives the optimal budget and allocation, accounting for spatial and serial dependence, spatial heterogeneity, neighborhood effects and sales dynamics.

To fill this gap, we formulate a spatiotemporal model that explicitly distinguishes national and regional advertising effects. While national advertising offers an efficient way to build sales globally, regional advertising enables managers to enhance sales locally. We capture observed dependencies across neighboring regions through neighborhood effects, and we capture unobserved dependencies across neighboring regions and across contiguous time periods through spatial correlation and serial correlation, respectively. Because of the unobserved spatial and serial dependencies, we obtain correlated multivariate Brownian motion affecting the sales

dynamics. Consequently, the resulting Hamilton–Jacobi–Bellman (HJB) equation is a second-order differential equation with multiple states (i.e., regional sales) and multiple controls (i.e., regional and national advertising expenditures). Nonetheless, we solve it analytically and thus derive the closed-form expressions for the optimal total budget and its regional allocations (see Propositions 1 and 2). This normative contribution not only furnishes the optimal budget and allocation simultaneously as recommended in the literature (see, e.g., Mantrala, Sinha, and Zoltners 1992) but also is novel to spatiotemporal literature in marketing and economics.

In addition to normative analysis, we extend Baltagi et al.'s (2007) estimation method by incorporating sales dynamics, neighborhood effects, and spatial heterogeneity. We then empirically validate the proposed model using market data from a leading German cosmetics company. The results indicate good fits for both in-sample and out-of-sample data. The coefficients for spatial and serial dependence are statistically significant. Using the estimated parameters, we evaluate the optimal budget and allocation. Comparing them with actual spends and BDI-based recommendations, we find misallocations at both the national and regional levels. The total budget should be reduced from €7.9 million to €5.9 million per month, and its split to national versus regional advertising should be changed from 92.4% to 85.9%. Furthermore, compared with actual profit, BDI-based allocations increase profit by approximately .37%, whereas the proposed approach increases profit by 5.07%. More important, it provides a systematic way to assess whether a specific region underspends (e.g., Region 2 should increase spending by 63%) or overspends (e.g., Region 7 should decrease spending by 83%). Such optimality assessments add diagnostic value for the chief marketing officer, because it indicates whether to shift the budget from national to regional advertising (or vice versa) and how to allocate resources across the regions.

We organize the rest of this article as follows: We first review the literature streams on BDI-based allocations, spatiotemporal models, and optimal allocation studies. Then, we propose a spatiotemporal model, derive normative results, and develop an estimation method. Next, we illustrate an empirical application and furnish substantive findings. We close by discussing the optimal allocations under time-varying parameters, the value of continuous-time analysis, the effects of discount rate, and pulsing versus even spending.

LITERATURE REVIEW

The BDI

The BDI indicates how a brand's sales perform relative to the size of the consumer market. Specifically, to obtain the BDI of a given region, we first determine (1) the brand sales in a given region as a ratio to the national sales and (2) the region's population as a ratio to the national population; then, BDI = 100 × [fraction in (1)/fraction in (2)]. Thus, a BDI score of 100 means that the region's sales are on par with the sales expected for its size, and scores above (below) 100 indicate over- (under-) performing regions. For example, a BDI score of 120 means that brand sales in that region are 20% greater than what would be expected from that region's market size; a BDI score of 75 points means

that its brand sales are 25% below the expected sales given its market size. A similar definition for category sales yields the category development index, which tends to be correlated with the BDI scores.

Brand managers prioritize various regions using the BDI scores. Advertising textbooks (see, e.g., Goodrich and Sissors 1996, p. 43; Hiebing and Cooper 2004, p. 247; Sissors and Baron 2002, p. 182) recommend allocating budgets proportional to the BDI scores. For example, consider a firm that operates in two regions, A and B, with sales fractions of .7 and .3, population shares of .4 and .6, and a regional budget of $1 million. Then, $BDI_A = 100 \times .7/.4 = 175$, and $BDI_B = 100 \times .3/.6 = 50$. So the resulting allocation to Region A = $(175/225) \times 1,000,000 = \$777,777$ and to Region B = $(50/225) \times 1,000,000 = \$222,223$. This example highlights that the BDI approach does not yield the spending on national advertising (and thus the total budget). Moreover, we do not know whether these allocations are optimal, because the optimal allocation depends on the sales lift resulting from the incremental ad spending (Abraham and Lodish 1990), whereas the BDI approach does not quantify this marginal effect of advertising. To this end, we need to formulate spatial models, which we review next.

Spatial and Spatiotemporal Models with Marketing Applications

Bradlow et al. (2005) and Bronnenberg (2005) provide extensive reviews of spatial models in marketing. In Table 1, we complement their works by comparing 19 studies on the following features: spatial heterogeneity, neighborhood effects, spatial dependence, serial dependence, sales dynamics, marketing decision variables, and optimal decisions.

Table 1 shows that most applications focus on pricing rather than advertising (e.g., Greenhut 1981; Jank and Kannan 2005; Pinske, Slade, and Brett 2002), while others examine spatial diffusion and demand patterns (e.g., Bell and Song 2007; Bronnenberg, Dhar, and Dubé 2007a; Duan and Mela 2009; Gatignon, Eliashberg, and Robertson 1989). Bhargava and Donthu's (1999) study is a notable exception for investigating the spatial effects of billboard effectiveness using field experiments. Moreover, most studies analyze spatial data for fast-moving consumer goods in food categories in the U.S. markets. In contrast, the current study analyzes spatial advertising data for a nonfood product category (cosmetics) in a major non-U.S. market (Germany).

Three types of spatial effects considered in the literature are spatial heterogeneity, neighborhood effects, and spatial dependence. Spatial heterogeneity occurs as a result of the nonuniform effects of space due to, for example, differences in urban growth, unequal populations, differential incomes, or differences in media consumption—all of which might result in different advertising effectiveness across regions (see Anselin 1988, pp. 11–15; Bradlow et al. 2005). To account for spatial heterogeneity, parameters differ across regions (i.e., we estimate region-specific parameters). Neighborhood effects arise because sales in a region depend not only on past sales in that region but also on past sales in neighboring regions. These effects can materialize as a result of various factors—for example, communication between people in different regions, cross-regional travel and business links, similar marketing programs and retail landscapes, and passive observation of products in other regions (Bell and Song 2007; Yang and Allenby 2003). Conversely, spatial dependency represents the covariation of observations across spatial units. It arises as a result of the effects of unobserved similarities between regions based on their socioeconomic makeup, usage of resources, or physical characteristics (Ter Hofstede, Wedel, and Steenkamp 2002). To account for spatial dependency, error terms are correlated across regions with a contiguity matrix.

Table 1 reveals that most studies incorporate neighborhood effects and spatial dependence. However, several studies do not allow response parameters to vary across regions (see the "Spatial Heterogeneity" column). As with spatial dependence, autocorrelation relaxes the assumption of independence among observations over time. Such serial dependence has been studied in Bronnenberg and Mela (2004), Bronnenberg and Sismeiro (2002), and Bronnenberg and Mahajan (2001). As for sales dynamics, diffusion models incorporate it by design (e.g., Bell and Song 2007). Bronnenberg and Sismeiro (2002) account for dynamic shocks in price effects, while Jank and Kannan's (2006) online learning choice model makes spatial predictions on the basis of spatially dispersed consumer choices observed up to the previous period. However, no study considers the five factors—spatial heterogeneity, neighborhood effects, spatial dependence, serial dependence, and sales dynamics—in the context of advertising.

Finally, we survey the type and scope of normative analyses. In the diffusion context, Choi, Hui, and Bell (2010) use numerical simulation to assess the impact of different imitation strategies but do not derive optimal decisions analytically. Similarly, Duan and Mela (2009) explore the role of spatial demand on outlet location and suggest desired locations for additional outlets numerically. Chan, Padmanabhan, and Seetharaman (2007) numerically investigate the impact of a potential merger between gasoline retail firms on the local markets. Thomadsen (2007) studies product positioning for asymmetric firms in spatial markets but computes equilibrium strategies numerically. Taken together, no study analytically derives optimal marketing strategies for dynamic and spatially related markets. Thus, this study augments the marketing literature.

Optimal Allocation of Advertising Budgets

We review a few prominent studies that offer normative advertising analyses; as we show subsequently, they ignore either neighborhood effects or spatial dependence or sales dynamics. Nerlove and Arrow's (1962) classic study envisions advertising to build a stock of goodwill that depreciates over time, and they obtain the dynamically optimal advertising strategy. We could generalize this solution for every region individually and then sum it up to determine the total advertising budget. However, the resulting budget would ignore the spatial effects (i.e., spatial heterogeneity, the neighborhood effects, and spatial dependence). Doyle and Saunders (1990) develop an approach for multiproduct advertising budgeting that accounts for advertising spillovers across multiple products. However, we cannot transfer their multiproduct solution to our multiregion setting, because products, unlike regions, do not exhibit spatial proximity (e.g., effects of neighboring regions). Skiera and Albers (1998) maximize profits obtained from multiple sales territories by optimizing travel costs. Although they incorporate

Table 1
SELECTED SPATIOTEMPORAL STUDIES IN MARKETING

Authors	Area	Spatial Units (Number of Units)	Spatial Heterogeneity	Neighborhood Effects	Spatial Dependence	Serial Dependence	Sales Dynamics	Marketing Decision Variables	Optimal Decisions
Gatignon, Eliashberg, and Robertson (1989)	Diffusion (international)	International (14 countries)	Yes				Yes		
Bronnenberg and Mela (2004)	Diffusion	National (95 U.S. markets)		Yes	Yes	Yes	Yes		
Albuquerque, Bronnenberg, and Corbett (2007)	Diffusion (international)	International (56 countries)	Yes	Yes	Yes		Yes		
Bell and Song (2007)	Diffusion	National (29,701 U.S. zip codes)		Yes	Yes		Yes		
Choi, Hui, and Bell (2010)	Diffusion	Regional (1459 Pennsylvanian zip codes)	Yes	Yes	Yes		Yes		Numerical
Bronnenberg and Sismeiro (2002)	Demand modeling	National (64 U.S. Information Resources Inc. markets)	Yes	Yes	Yes	Yes	Yes	Price	
Duan and Mela (2009)	Demand modeling	Local (continuous city grid)		Yes	Yes			Price	Numerical
Ter Hofstede, Wedel, and Steenkamp (2002)	Segmentation	International (120 European regions)	Yes	Yes	Yes			Positioning/store image	
Yang and Allenby (2003)	Consumer/choice behavior	Local (857 local U.S. consumers)			Yes				
Mittal, Kamakura, and Govind (2004)	Customer satisfaction	National (164k U.S. consumers)	Yes		Yes			Service/quality	
Jank and Kannan (2006)	Targeting	National (160 consumers across U.S. zip codes)	Yes		Yes			Price	
Jank and Kannan (2005)	Targeting and pricing	National (160 consumers across US zip codes)	Yes		Yes		Yes	Price	
Greenhut (1981)	Pricing	International (3 countries)	Yes					Price explained	
Case (1991)	Pricing	Regional (8 Bali districts)		Yes	Yes			Price	
Pinske, Slade, and Brett (2002)	Pricing	National (305 terminals in 48 U.S. states)		Yes	Yes			Price	
Chan, Padmanabhan, and Seetharaman (2007)	Pricing	Local (1550 continuous grid points in Singapore)	Yes					Price	Numerical
Thomadsen (2007)	Pricing	Local (U.S. county; simulation)	Yes					Price	Numerical
Bronnenberg and Mahajan (2001)	Pricing and promotion	National (64 U.S. Information Resources Inc. markets)	Yes	Yes	Yes	Yes		Price promotion	
Bhargava and Donthu (1999)	Advertising	Local (U.S. city districts)	Yes	Yes				Advertising	
The current study	Advertising	National (7 German regions and national level)	Yes	Yes	Yes	Yes	Yes	Advertising	Analytical optimal allocation for the total and regional budgets

spatial heterogeneity in sales potential, they ignore sales dynamics, neighborhood effects, and spatial dependence. More recently, Naik and Raman (2003) investigate the allocation for multimedia advertising in a dynamic setting and prove a counterintuitive result: As cross-media synergy increases, managers should increase the total budget, and this incremental budget should be allocated in inverse proportion to the media effectiveness—that is, the less (more) effective medium gets the larger (smaller) share of the incremental budget. Akin to Doyle and Saunders (1990), the multiple media do not exhibit spatial proximity, and so Naik and Raman also ignore the spatial and neighborhood effects. Thus, there is a gap in the extant literature on spatiotemporal allocation of national and regional advertising budgets. To fill this gap, we formulate a dynamic advertising model with spatial effects and then address budget allocation and model estimation.

MODEL FORMULATION

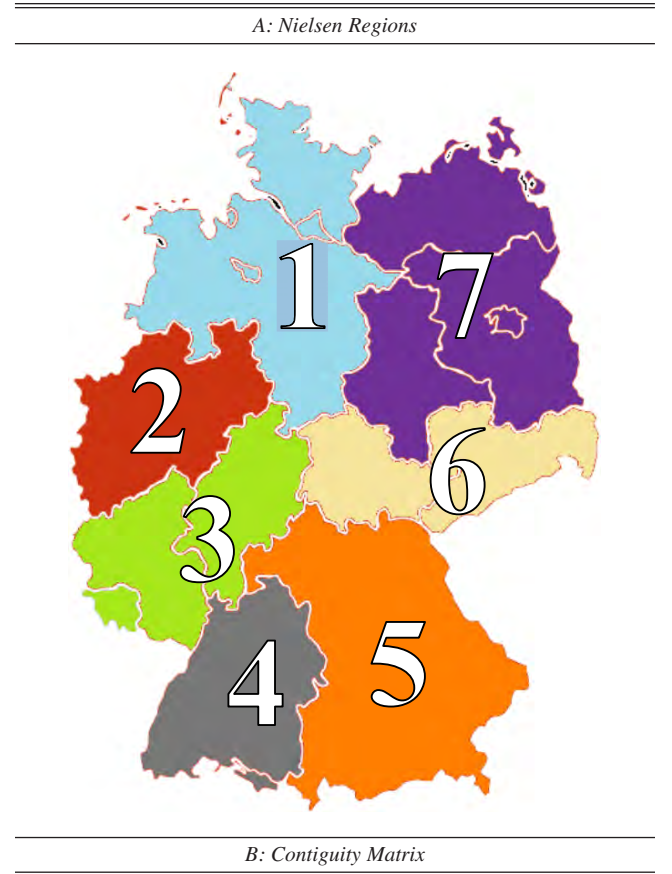### Sales Dynamics, Neighborhood Effects, and Spatial Heterogeneity

Marketing research has shown that consumers' response varies across both time (e.g., Naik and Raman 2003) and regional markets (e.g., Bhargava and Donthu 1999; Lodish 2007) and as a result of interregional effects (Bell and Song 2007). Thus, we incorporate sales dynamics through carryover effects, neighborhood effects through the impact of lagged sales in neighboring regions, and spatial heterogeneity through region-specific parameters. For each region i, the local region's sales depend on its own and neighboring regions' lagged sales as well as both the regional and national advertising, which we express as follows:

$$(1) \quad S_{it} = \lambda_i S_{i,t-1} + \beta_i \sqrt{R_{it}} + \alpha_i \sqrt{N_t} + \sum_{\substack{j=1 \\ j \neq i}}^{K} \gamma_{ij} S_{j,t-1} + \varepsilon_{it},$$

$$i, j = 1, ..., K, \quad t = 1, ..., T,$$

where S_{it} measures the units sold in region i at time t, S_{jt-1} denotes the sales in neighboring region j at time t – 1, R_{it} and N_t are the regional and national dollars spent on advertising, and the error terms $\varepsilon_t = (\varepsilon_{1t}, ..., \varepsilon_{Kt})' \sim N(0, \Sigma_\varepsilon)$. The region-specific parameters λ_i are the regional carryover effects, γ_{ij} quantifies the spatial effects of neighboring region j's lagged sales, β_i measures regional ad effectiveness, and α_i measures the effect of national advertising on local sales. Because the marketing mix differs between regional (direct mail, local newspapers, radio) and national (television, magazines, national newspapers) advertising, the respective advertising response effects (β_i and α_i) also vary. To capture spatial effects, we specify interregional dependence between contiguous regions through the contiguity matrix $\tilde{C}$, whose elements $\tilde{c}_{ij}$ equal 1 if the region i shares its border with the region j and 0 if otherwise (see Panel A of Figure 2 for the seven Nielsen regions of Germany and Panel B for the corresponding contiguity matrix $\tilde{C}$). We standardize this contiguity matrix such that each row sums to unity and denote the resulting matrix as C (and its elements by c_{ij}). Thus, the neighborhood effects γ_{ij} equal γc_{ij}, where γ represents the overall neighborhood effect. Consistent with the prior literature (e.g., Chintagunta 1993),

Figure 2
SEVEN NIELSEN REGIONS AND THEIR CONTIGUITY MATRIX

A: Nielsen Regions

B: Contiguity Matrix

$$\tilde{C} = \begin{bmatrix} \textbf{Regions} & \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} & \textbf{6} & \textbf{7} \\ \textbf{1} & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ \textbf{2} & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \textbf{3} & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ \textbf{4} & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \textbf{5} & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \textbf{6} & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \textbf{7} & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

the square roots in Equation 1 incorporate the notion of diminishing returns, which means that, at a certain point, incremental sales decrease with increased ad spending. The next subsection introduces spatial and serial dependence in the error term ε_{it}.

Spatial and Serial Dependence

Several factors not explicitly included in the model, and thus relegated to the error terms, introduce spatial dependence (Bradlow et al. 2005; Bronnenberg 2005; Chintagunta, Dubé, and Goh 2005). Such unobserved factors emerge because neighboring "regions ... often share climate,

resources, history and sociodemographic and economic makeup" (Ter Hofstede, Wedel, and Steenkamp 2002, p. 161). Because these factors do not explicitly enter the model, we allow the error terms to be correlated across regions. Given the standardized contiguity matrix C, we express the error term ε_{it} as follows:

$$(2) \qquad \varepsilon_{it} = \mu \sum_{j=1}^{K} c_{ij}\varepsilon_{jt} + \eta_{it},$$

where i and j denote the various regions, μ is the spatial correlation, and the error vector $\eta_t = (\eta_{1t}, \ldots, \eta_{Kt})' \sim N(0, \Sigma_\eta)$. Equation 2 allows for neighboring regions' shock to affect the focal region. Because this shock does not persist over time, we extend the error structure to incorporate serial dependence.

Spatial dependence reflects disturbances between regions; serial dependence captures shocks within a region over time. That is, the unobserved shocks within a region may carry over to subsequent periods (e.g., Bronnenberg and Mahajan 2001; Bronnenberg and Mela 2004). Such shocks could occur as a result of unobserved consumer or manufacturer behavior or actions by other unobserved participants, such as distributors or retailers (Bronnenberg and Sismeiro 2002). To incorporate such serial dependence, we modify the error structure in Equation 2 by allowing η_{it} to be serially correlated with $\eta_{i, t-1}$ as follows:

$$(3) \qquad \eta_{it} = \omega\eta_{i, t-1} + v_{it},$$

where ω represents the serial dependence and $v_t = (v_{1t}, \ldots, v_{Kt})' \sim N(0, \Sigma_v)$. In the next section, we derive closed-form expressions for optimal advertising budget, its optimal split between national and regional spends, and its optimal allocation across multiple regions.

SPATIOTEMPORAL ALLOCATION

Continuous-Time Dynamics and Uncertainty

To facilitate the derivation of optimal budget and allocations, we convert the discrete-time model to its continuous-time analog (see Malliaris and Brock 1982, pp. 66–68). Specifically, we rewrite Equation 1 as follows:

$$dS_i(t) = [-\delta_i S_i(t) + \beta_i\sqrt{R_i(t)} + \alpha_i\sqrt{N(t)}$$
$$+ \sum_{\substack{j=1 \\ j \neq i}}^{K} \gamma_{ij}S_j(t)]dt + \sum_{j=1}^{K} \tilde{\sigma}_{ij}d\tilde{W}_j(t),$$

where $\delta_i = (1 - \lambda_i)$ and $d\tilde{W}_j$ is a random process whose properties we derive in the Web Appendix (see http://www.marketingpower.com/jmr_webappendix) to characterize the explicit dependence of $\tilde{\sigma}_{ij}$ on the spatial and serial dependence parameters (μ and ω). Thus, denoting

$$f_i = -\delta_i S_i + \beta_i\sqrt{R_i} + \alpha_i\sqrt{N} + \sum_{\substack{j=1 \\ j \neq i}}^{K} \gamma_{ij}S_j$$

for i = 1, …, K, the system of regional continuous-time stochastic sales differential equations is

$$(4) \qquad dS_i = f_i dt + \sum_{j=1}^{K} \tilde{\sigma}_{ij}d\tilde{W}_j.$$

Long-Term Future Profit Expectation

Next, to evaluate the long-term future profit resulting from Equation 4, we let ρ be the discount rate and m_i represent the margins from the regions i. Then, the long-term expected future profit is

$$(5) \quad J(R_1, \ldots, R_K, N) = E\left[\int_0^{\infty} e^{-\rho t}\left(\sum_{i=1}^{K} m_i S_i - \sum_{i=1}^{K} R_i - N\right)dt\right],$$

where $E(\cdot)$ denotes the expectation and the integral sums up the discounted future operating profit, which equals the gross profit less the total ad spends (shown in the parentheses).

In Equation 5, the expected profit $J(\cdot)$ depends on the spending levels R_i and N. A meager ad spending (R_i, N) would generate limited sales and earn small profit; as (R_i, N) increases, sales S_i increase, in turn enhancing profit J; however, beyond a certain level, further advertising increases sales but with a diminishing rate, and thus profit J decreases. Given this inverted U shape of $J(\cdot)$ with respect to its arguments, managers should operate at the "sweet spot," where the regional and national advertising are neither too little nor too much. To this end, they should find the optimal regional advertising R_i^* and the optimal national advertising N^* by maximizing the long-term future profit in Equation 5 subject to the stochastic dynamic evolution in Equation 4, which incorporates sales dynamics, neighborhood effects, spatial heterogeneity, and spatial and serial dependence.

Optimal Budget, Allocations, and Split

To maximize $J(R_1, \ldots, R_K, N)$, we define the value function, which represents the largest attainable profit when we optimally set all the regional and national advertising spends. Let the value function $V(S_1, \ldots, S_K) = \text{Max}[J(R_1^*, \ldots, R_K^*, N^*)]$. Then, the value function satisfies the stochastic HJB equation:

$$(6) \quad \rho V = \text{Max}\left[\underbrace{\left(\sum_i m_i S_i - \sum_i R_i - N\right)}_{\substack{\text{Profit from today's} \\ \text{allocation decision}}} + \underbrace{\sum_i \dot{V}_i f_i}_{\substack{\text{Profit from} \\ \text{future impact} \\ \text{of today's} \\ \text{allocation} \\ \text{decision}}} + \underbrace{\tfrac{1}{2}\text{Tr}\left(\ddot{V}\Sigma_\varepsilon\right)}_{\substack{\text{Profit from} \\ \text{spatial and} \\ \text{serial} \\ \text{dependence}}}\right],$$

where $\dot{V}_i = \partial V/\partial S_i$, the K × K matrix $\ddot{V}$ consists of the elements $\{v_{ij}\}$, $v_{ij} = \partial^2 V/\partial S_i\partial S_j$ and $\text{Tr}(\cdot)$ denotes the trace of a matrix.

Equation 6 is a second-order differential equation for the value function V, and it constitutes the necessary and sufficient condition for an optimum (Lewis 1986, p. 298; Sethi and Thompson 2000, pp. 345–47). Its first term on the right-hand side captures the immediate profit from the current states (i.e., sales) and controls (i.e., national and regional advertising). The second term incorporates the change in future profit due to changes in sales trajectories induced by current advertising decisions. The last term reflects the

profit consequences arising from the spatial and serial dependence.

By analytically solving the stochastic HJB Equation 6 in the Web Appendix (see http://www.marketingpower.com/jmr_webappendix), we obtain the closed-form expressions for the optimal regional and national spends R_i^* and N^*, which we present next.

P$_1$: For each region i, the optimal regional advertising expenditure is

(7) $$R_i^* = \tfrac{1}{4}(\beta_i b_i)^2;$$

the optimal national advertising expenditure is given by

(8) $$N^* = \tfrac{1}{4}(\alpha_1 b_1 + ... + \alpha_K b_K)^2, \text{ where}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \end{bmatrix} = \begin{bmatrix} \rho + \delta_1 & -\gamma_{21} & \cdots & -\gamma_{K1} \\ -\gamma_{12} & \rho + \delta_2 & \cdots & -\gamma_{K2} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1K} & -\gamma_{2K} & \cdots & \rho + \delta_K \end{bmatrix}^{-1} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_K \end{bmatrix}.$$

Thus, the optimal total B^* and its optimal split ϕ^* are given by

(9) $$B^* = N^* + \sum_{i=1}^{K} R_i^*, \text{ and } \phi^* = N^*/B^*.$$

(For the proof, see the Web Appendix at http://www.marketingpower.com/jmr_webappendix.)

We note that the optimal allocations in Equations 7 and 8 depend on the neighborhood effects γ_{ij} through b_i. Remarkably, the optimal allocations do not depend on the spatial and serial correlations (μ and ω), because managers do not control the uncertainty in realized sales due to spatial and serial effects. Nonetheless, the realized sales influence the estimated parameter values and their efficiency (see Equations 12 and 13). Thus, the effects of spatial and serial correlations manifest in the estimated allocations and their operating range.

This discussion completes the optimal budgeting and allocation of dynamic spatiotemporal models. To apply the preceding formulae in practice, managers need to obtain parameter values by estimating the model Equations 1–3 using market data. Thus, we develop an estimation method in the next section.

MODEL ESTIMATION

Recently, Baltagi et al. (2007) provided a method that accounts for spatial and serial dependence. Their method groups observations of all K regions together and then stacks these groups for t = 1, …, T time periods (Baltagi et al. 2007, p. 7). This way of stacking is not appropriate when estimating the proposed model for three reasons. First, to accommodate sales dynamics, we need to keep contiguous time periods grouped together. Second, to incorporate spatial heterogeneity, we need to create a block (rather than stacked) diagonal structure for the regressor matrix. Third, to include neighborhood effects, we stack the regressor

matrix with a column formed by composite variables of lagged neighboring region sales. These features result in a new error covariance matrix that differs from Equation 2.12 in Baltagi et al. To obtain the proper error covariance matrix, we construct new vectors and matrices.

Let $\theta_i = (\lambda_i, \beta_i, \alpha_i)'$ denote the parameters for each region i, and $Y_i = (S_{i2}, ..., S_{iT})'$ be the $(T-1) \times 1$ vector of sales starting from the period 2 through T (because the lag operation creates a missing value) for each region i. We then create a matrix X_i of dimension $(T-1) \times 3$ by stacking $(S_{i,t-1}, R_{it}, N_{it})'$ from t = 2, …, T. We generate a composite vector $\tilde{X}_i = (\tilde{X}_{i2}, ..., \tilde{X}_{iT})'$ of length $(T-1)$, where $\tilde{X}_{it} = \Sigma_{j=1:j\neq i}^{K} c_{ij} S_{j,t-1}$ for t = 2, …, T. Using these vectors and matrices, we convert Equation 1 into the following system:

(10) $$\begin{bmatrix} Y_1 \\ \vdots \\ Y_K \end{bmatrix} = \begin{bmatrix} X_1 & 0 & \cdots & \tilde{X}_1 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \cdots & X_K & \tilde{X}_K \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_K \\ \gamma \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_K \end{bmatrix},$$

where $\varepsilon_i = (\varepsilon_{i2}, ..., \varepsilon_{iT})'$. Thus, we re-express the preceding equation as follows:

(11) $$Y = X\theta + \varepsilon,$$

where $Y = (Y_1', ..., Y_K')'$, $\theta = (\theta_1', ..., \theta_K', \gamma)'$, $\varepsilon = (\varepsilon_1', ..., \varepsilon_K')'$, and the matrix X constructed as shown in Equation 10.

For parameter estimation, we use Equations 12 and 13:

(12) $$\hat{\theta} = (X'\tilde{\Sigma}_\varepsilon^{-1}X)^{-1}X'\tilde{\Sigma}_\varepsilon^{-1}Y, \text{ and}$$

(13) $$\tilde{\Sigma}_\varepsilon(\mu, \omega) = [(I - \mu C)^{-1}\Sigma_v(I - \mu C)^{-1'} \otimes M(\omega)],$$

where $$M(\omega) = \frac{1}{(1-\omega^2)} \begin{bmatrix} 1 & \omega & \omega^2 & \cdots & \omega^{T-2} \\ \omega & 1 & \omega & \cdots & \omega^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{T-2} & \omega^{T-3} & \omega^{T-4} & \cdots & 1 \end{bmatrix},$$

and $\otimes$ indicates a Kronecker product. We derive Equation 13 in the Web Appendix (see http://www.marketingpower.com/jmr_webappendix). For statistical inference, we obtain the standard errors using the square root of the diagonal of the matrix $(X'\tilde{\Sigma}_\varepsilon^{-1}X)^{-1}$. For robust inference, we obtain standard errors from the matrix $(X'\tilde{\Sigma}_\varepsilon^{-1}X)^{-1}[X'\tilde{\Sigma}_\varepsilon^{-1}\Psi\tilde{\Sigma}_\varepsilon^{-1}X](X'\tilde{\Sigma}_\varepsilon^{-1}X)^{-1}$, where Ψ is a diagonal matrix of squared residuals (for details, see MacKinnon and White 1985). For spatiotemporal estimation, we maximize the concentrated log-likelihood function $LL(\mu, \omega) = .5[Ln(|\tilde{\Sigma}_\varepsilon|) + \hat{\varepsilon}'\tilde{\Sigma}_\varepsilon^{-1}\hat{\varepsilon}]$ with respect to (μ, ω), where $\hat{\varepsilon} = Y - X\hat{\theta}$. In summary, we extend Baltagi et al.'s (2007) work by enabling maximum likelihood estimation and inference of dynamic spatiotemporal models.

EMPIRICAL ANALYSIS

In this section, we describe the data, estimation results, and allocation insights. We also perform robustness checks and cross-validation to ensure the validity of findings.

Data

The German cosmetics market, approximately €5.4 billion in size, consists of three equally sized segments: decorative cosmetics, face care, and body care (Nielsen 2008).

We focus on the decorative cosmetics segment and analyze the proprietary data from the dominant market leader, whose identity remains confidential. The participating company operates in all seven Nielsen regions of Germany shown in Figure 2, Panel A. A Nielsen region aggregates areas that exhibit substantial similarity on various metrics (e.g., consumption patterns, demographics, psychographics, purchasing power) and differ sufficiently with other Nielsen regions. Consequently, a Nielsen region need not adhere to political or provincial boundaries; for example, Region 6 combines Saxony and Thuringia. Brand sales and advertising spending, respectively, were 200 million units and €214 million over 29 months. On a monthly basis, the coefficient of variation (i.e., ratio of the standard deviation to the mean) for brand sales is 14%, and for advertising, it is 45%, whereas market shares and prices fluctuate by less than 1% and 3%, respectively. Given the lack of variation in shares and prices over time, we focus on sales and advertising. National advertising (e.g., television, magazines, national newspapers) consumes more than 90% of the total budget; the rest, allocated to the seven regions, is spent on direct mail, local newspapers, and radio. Although the regional ad spends are small, the coefficient of variation is more than 100%, compared with 45% for the national spends. Table 2 summarizes the descriptive statistics.

Estimation Results

Robustness checks. To account for endogeneity of advertising, we apply the instrumental variables approach (e.g., Bronnenberg and Mahajan 2001, p. 286). At the regional level, we predict each region's ad spends using spending in noncontiguous regions (i.e., neighbors of neighbors). At the national level, we predict national ad spends using two-period lagged national advertising. Because of the two-period spatial and temporal lags, we mitigate the correlations between the instruments and the spatial and serial error components (which have one-period dependence). In the Web Appendix (see http://www.marketingpower.com/jmr_webappendix), using the Engle–Hendry–Richard test, we present evidence that the resultant instruments exhibit not only high goodness of fit but also weak exogeneity. Weak exogeneity means that, when we factorize a joint density of sales and advertising, g(S, R, N), into the conditional density of sales given advertising g(S|R, N) and the marginal density of advertising g(R, N), the precise specification of the marginal density is not relevant, and the model

estimation using only the conditional density entails no loss of information.

In addition, we test for parameter constancy using the Cusum test developed by Brown, Durbin, and Evans (1975) and extended to dynamic models by Ploberger and Kramer (1992).[1] If true parameters vary over time but the proposed model assumes constancy, the model fit worsens, and its residuals become large over time. Thus, the cumulative sum of residuals meanders away from mean zero and crosses the confidence bounds. The Cusum test results for our data indicate that the cumulative residuals lie within the confidence bounds, providing empirical support for parametric invariance. Given weak exogeneity and parametric invariance, according to Ericson and Irons (1994), these instruments are superexogenous (Ericson and Irons 1994, p. 14).

Next, we compare the proposed model with other specifications, namely, the S-shaped and log-advertising response model as well as the log-log model. We find that it outperforms the S-shaped model on the bias-corrected Akaike information criterion (AIC_C). The log-advertising response model and the log-log model result in negative carryover rates and negative regional advertising effects, respectively; thus, we reject these specifications because they lack face validity. We retain the proposed model because it enjoys stronger empirical support. We also compare the proposed model with one that includes intercepts. The results indicate that the proposed model performs better than the model with intercepts on both the information criterion (AIC_C = 28,342 for the proposed model vs. 28,357 for the model with intercepts) and the likelihood ratio test (test statistic = 5.36, which does not exceed the critical value χ^2_7 = 14.07), thus rejecting the need for intercepts.

Finally, to verify whether regional and national spends are substitutes, we include advertising substitution effects in the sales model in Equation 1.[2] We study two types of substitution effects: pure substitution and nested substitution. We model pure substitution effects by replacing $\beta_i\sqrt{R_{it}} + \alpha_i\sqrt{N_t}$ in Equation 1 with a single advertising variable $\beta_i\sqrt{R_{it} + N_t}$ for each region and then estimate this specification. The resultant AIC_C value of 28,350 exceeds the AIC_C value of 28,342 for the proposed model. Because this difference exceeds 2 points (Burnham and Anderson 2002, p. 70), the proposed model enjoys stronger empirical support. We test nested substitution by extending Equation 1 with the term $\psi_i\sqrt{R_iN}$ for each region. This specification nests the possibility of regional and national advertising being complements ($\psi_i > 0$), substitutes ($\psi_i < 0$), or independent ($\psi_i = 0$) (see Ingene and Parry 1995, p. 1195). We estimate this extended model and obtain an AIC_C value of 28,359, which is higher than that for the proposed model. Thus, the proposed model receives stronger empirical support (for insignificant values of $\hat{\psi}_i \approx 0$, see the Web Appendix at http://www.marketingpower.com/jmr_webappendix).

Model fit and forecasts. Table 3 shows that the system-wide R-square is approximately 98%, indicating a good model fit to the in-sample data. To assess out-of-sample forecasting performance, we conduct cross-validation by fitting the model using the first 20 months of data and using the last 9 months as the holdout sample. For each of the

Table 2
DESCRIPTIVE STATISTICS

Regions	Units Sold per Month		Ad Spends per Month (€)	
	Average	SD	Average	SD
1	1,250,554	168,392	112,660	109,202
2	1,555,474	216,659	160,823	172,363
3	1,083,338	146,819	62,346	61,532
4	996,878	138,016	61,602	59,958
5	1,204,748	163,440	72,663	60,520
6	394,853	53,768	43,836	55,844
7	663,744	93,272	93,136	114,161
National	7,149,589	968,735	7,153,914	3,291,509

Notes: Sample size T = 29 months.

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[1]We thank an anonymous reviewer for this suggestion.
[2]We thank an anonymous reviewer for this suggestion.

seven regions and the national level, the cross-validation R-square values vary from 97.12% to 98.06%, with the median of 97.84%. Thus, both the in-sample model fit and out-of-sample forecasts are satisfactory.

Sales dynamics, neighborhood effects, and spatial heterogeneity. Table 3 also shows that the estimated carryover effects and advertising effectiveness vary across the seven regions. We note that all the signs for the estimated effects are positive, as it should be. The robust t-values indicate that, in each region, both ad spending and lagged sales have significant effects.

The estimated neighborhood effect $\hat{\gamma}$ equals .1065. To test its significance jointly with all other parameters, we conduct the likelihood-ratio test. The test statistic of 32.02 exceeds the critical value $\chi_1^2 = 3.84$, thus favoring the inclusion of neighborhood effects in the model. This finding comports with the corrected AIC. Substantively, the neighboring regions exert a positive impact on focal region's sales. On average, 10.65% of the composite sales from neighboring regions spills over into the focal region. The elasticity of the various neighborhood effects equal (.0781, .0793, .1054, .1214, .0725, .2811, .1309)′, which lends support to the presence of spatial heterogeneity. The large elasticities arise in regions with high economic integration and business orientation. For example, Region 6 is commercially integrated with both Bavaria to its southwest and Berlin to its north.

Using the coefficient of variation, we assess the extent of spatial heterogeneity. We find that the coefficient of variation for carryover effects is approximately 11%. Consequently, spatial heterogeneity for the carryover effects exists, but its magnitude is small. In contrast, the coefficients of variation for regional and national advertising are 39.6% and 37.1%, respectively, revealing a large spatial heterogeneity in advertising effectiveness. That is, as we noted previously, market data lend support to the presence of spatial heterogeneity. Across Germany, regional differences in media usage and advertising effectiveness reflect the variation in these estimates. In addition, Columns 5 and 6 of Table 3 provide the regional and national advertising

Table 3
PARAMETER ESTIMATES

Regions	Lagged Sales ($\hat{\lambda}_i$)	Regional Advertising ($\hat{\beta}_i$)	National Advertising ($\hat{\alpha}_i$)	Regional Elasticity	National Elasticity
1	.6384 (14.20)	109.01 (2.58)	135.82 (5.30)	.01	.14
2	.6570 (14.56)	143.31 (2.89)	152.13 (4.61)	.02	.13
3	.5702 (10.77)	136.17 (1.90)	134.83 (6.23)	.01	.16
4	.5720 (9.70)	123.54 (2.25)	116.70 (5.57)	.01	.15
5	.6080 (14.00)	157.77 (2.56)	145.34 (6.08)	.02	.16
6	.4617 (3.10)	41.01 (1.97)	39.52 (4.95)	.01	.13
7	.5878 (9.31)	61.02 (3.09)	71.83 (4.98)	.01	.14
Neighborhood effect			.1065 (SE = .065)		
Spatial dependence			.0403 (SE = .019)		
Serial dependence			−.3004 (SE = .044)		
Systemwide R^2			98.45%		

Notes: t-values are in parentheses.

elasticities, which show that the coefficient of variation for regional ad elasticities is 19.2% and for national ad elasticities is 9.1%. This substantial variation in regional ad elasticities emphasizes the importance of spatial heterogeneity. Finally, we observe in Table 3 that the regional elasticities are an order of magnitude smaller than the national elasticities. Theoretically, this finding follows from the fact that the ratio of national to regional elasticities is proportional to the number of regions (under symmetric regions). Empirically, national advertising achieves greater reach than regional advertising, which the firm employs to enrich its media mix.

Spatial and serial dependence. The spatial correlation $\hat{\mu}$ is positive and significant at the 95% confidence level. Specifically, the maximum likelihood estimate of $\hat{\mu} = .0403$ (SE = .019). Given this small but significant magnitude of $\hat{\mu}$, we test its stability using alternative contiguity matrices. We find that $\hat{\mu} = .0400$ (SE = .019) when the contiguity matrix is based on relative mean age across regions, $\hat{\mu} = .0402$ (SE = .019) when it is defined on relative female to male ratio, and $\hat{\mu} = .0464$ (SE = .026) when it is defined on relative population density. These results enhance our confidence in the finding of positive spatial dependence, which means positive (negative) shocks within a region increase (decrease) the sales of immediate neighbors. Because lower values of spatial dependence imply greater regional heterogeneity (Ataman, Mela, and Van Heerde 2007, p. 19), a small value of $\hat{\mu}$ emerges partly due to Nielsen's way of combining regions that enhances interregion heterogeneity.

The serial dependence $\hat{\omega}$ is negative and significant at the 95% confidence level. Specifically, the maximum likelihood estimate of $\hat{\omega} = -.3004$ (SE = .0441). This finding reveals that in addition to the spatial effects on the neighboring regions, shocks within a region alternate (in sign) over subsequent periods. A few plausible reasons include stocking behavior of consumers and retailers (Hall 1988; McGuire 1977), lag structure (Rao 1986), and distributors' inventory management decisions (Baganha and Cohen 1998; Ramey 1991). Given that the magnitude of $\hat{\omega}$ is approximately one-third, such oscillatory shocks lasts for three and a half months because about a third of it dissipates every month. On the basis of these estimation results, we next address the substantive questions: how much to spend optimally on advertising, how much of it should be set aside for national advertising, and how to optimally allocate the rest to the seven Nielsen regions of Germany.

Allocation Insights

BDI versus optimal versus actual allocations. As illustrated in the example in literature review, we compute the BDI scores for the seven regions and the resulting BDI-based allocations (see Table 4, Panel A). Recall the two drawbacks of the BDI approach: (1) The national budget cannot be determined, and (2) the optimality of these allocations cannot be ascertained because of its model-free nature. To overcome the second drawback, we use the fitted sales model, thus extending the standard BDI approach. Equation 1 provides the sales model, which suggests that long-term sales $\tilde{S} = D^{-1}\tilde{A}$, where $\tilde{S} = (\tilde{S}_1, ..., \tilde{S}_K)'$, $\tilde{A} = (\tilde{A}_1, ..., \tilde{A}_K)'$, $\tilde{A}_i = (\beta_i\sqrt{\tilde{R}_i} + \alpha_i\sqrt{N})$, and

Table 4

MONTHLY ALLOCATIONS AND ANNUAL PROFIT

A: BDI-Based Results

Regions	BDI	BDI Allocations[a] $\widetilde{R}_i$ (€)	Long-Term Sales $\widetilde{S}_i$ (Units)	Margin m_i (€)	Revenues $m_i \times \widetilde{S}_i$ (€)
1	109	94,847	1,415,325	1.83	2,590,045
2	99	86,636	1,734,002	1.81	3,138,544
3	112	97,447	1,250,012	1.76	2,200,021
4	107	93,052	1,151,795	1.61	1,854,390
5	111	96,679	1,384,851	1.81	2,506,580
6	70	60,742	454,395	1.18	536,186
7	76	66,231	750,120	1.20	900,144
National	—		Cannot be computed		
National ad spend (from data)		7,289,534			
Totals (monthly)		7,885,168			13,725,910
Annual BDI profits			€70,088,902		

B: Optimal and Actual Results

Regions	Optimal Allocation[b] R_i^*, N_i^* (€)	Optimal Long-Term Sales S_i^* (Units)	Optimal Revenues $m_i \times S_i^*$ (€)	Actual Allocations[c] R_i, N_i (€)	Actual Long-Term Sales S_i^* (Units)	Actual Revenues $m_i \times S_i^*$ (€)
1	122,778	1,225,541	2,242,740	111,399	1,424,366	2,606,590
2	253,082	1,573,827	2,848,627	155,679	1,774,272	3,211,432
3	134,541	1,094,709	1,926,689	62,297	1,230,650	2,165,944
4	98,682	998,805	1,608,076	61,614	1,130,397	1,819,940
5	209,225	1,246,952	2,256,983	71,756	1,364,032	2,468,897
6	4,483	378,307	446,402	43,008	456,625	538,818
7	15,460	619,922	743,906	89,882	744,178	893,014
National	5,120,586			7,289,534		
Totals	5,958,837		12,073,423	7,885,169	8,054,711	13,704,635
Annual profits	€73,375,032			€69,833,592		
Split	85.93%			92.44%		

[a]$\widetilde{R}_i \propto$ BDI/100 × Total Regional Budget.

[b]Based on P_1, parameter values in Table 3, and margins in Panel A of Table 4.

[c]Average spending based on 28 months.

$$D = \begin{bmatrix} \delta_1 & -\gamma_{21} & \cdots & -\gamma_{K1} \\ -\gamma_{12} & \delta_2 & \cdots & -\gamma_{K2} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1K} & -\gamma_{2K} & \cdots & \delta_K \end{bmatrix}.$$

Using the parameter estimates from Table 3 and margin information from Table 4 (see Column 5 in Panel A), we then compute the regional long-term sales (see Column 4) and revenues (see Column 6). However, because the national spending is undeterminable (because of the first drawback), we must use the actual national spending. Then, the resulting annual BDI profit, which equals total revenues less the overall ad spends, is €70,088,902 (see Panel A of Table 4)—an increase of .37% over the actual profit.

Can managers achieve a higher profit? To this end, in Table 4, we compute the normative ad spends using Equations 7 and 8. We obtain the long-term sales $S^* = D^{-1}A^*$, where $S^* = (S_1^*, ..., S_K^*)'$, $A^* = (A_1^*, ..., A_K^*)'$, and $A_i^* = (\beta_i \sqrt{R_i^*} + \alpha_i \sqrt{N^*})$. The corresponding revenues appear in Table 4, Panel B (Columns 3 and 4). Then, the resultant annual optimal profit is €73,375,032, representing 4.7% increase over the BDI approach.

How do both the approaches compare with managers' actual decisions? Because the actual annual profits were €69,833,592 (for details, see Table 4, Panel B), the BDI approach yields a .37% profit increase. In contrast, by using

the proposed method, first, managers can earn a higher profit (~5.07%) than by using their own actions. Second, they learn about the optimal overall budget, which the BDI approach does not indicate. By knowing the optimal overall budget, they discover whether they are over- or underspending, which is unknowable in the absence of a benchmark. Third, they ascertain the optimal split between national and regional spends. Specifically, the actual split was 92.4%, while the optimal split should be $\hat{\phi}^* = 85.9\%$ (see Equation 9 and Table 4, Panel B). Thus, the proposed method yields a larger profit and reveals budget misallocations, whereas the BDI leads to meager profit improvement. Furthermore, we make profit comparisons with and without neighborhood effects and spatial dependence, revealing that profit increase is reduced by 22.68% when these effects are ignored. Together, the results further emphasize the importance of spatial effects in marketing (e.g., Bell and Song 2007; Bronnenberg and Mahajan 2001; Bronnenberg and Mela 2004).

Budget misallocations. Many large companies misallocate resources; for example, Corstjens and Merrihue (2003, p. 8) interview senior executives from 20 leading global companies and find "widespread frustration on the matter [of misallocations]. Many complained that determining where and how marketing budgets should be allocated … seemed virtually impossible." Our analysis comports with their findings, and the participating company also seems to be misallocating

advertising dollars. Specifically, Figure 3 reveals region-specific misallocations and profit consequences.

Figure 3 identifies overspending in Regions 6 and 7 (i.e., the dot is above the dash) and underspending in Regions 2, 3, 4, and 5 (i.e., the dot is below the dash). It also indicates the magnitudes of misallocations. For example, in Regions 6 and 7, the company overspends by approximately 89.5% and 82.8%, respectively, whereas in Regions 2, 3, 4, and 5, the magnitudes of underspending are 62.5%, 115.9%, 60.1%, and 191.5%, respectively. Although managers seem to be operating nearly optimally in Region 1, the model shows advertising budget should be increased by 10.2%.

We emphasize that the changes in allocations are proportional to neither the regional sales nor per capita regional sales, as the BDI approach suggests. For example, Regions 1 and 2 have higher sales than Regions 3 and 5, and yet the optimal allocation procedure recommends larger increases for Regions 3 and 5.

In addition to region-specific knowledge, the figure highlights the misallocation in the national budget (~29.7%) and, consequently, the total budget (~24.4%). In the absence of a method for optimal allocation, which provides the normative benchmarks, it is indeed virtually impossible—as Corstjens and Merrihue (2003) note—to know which regions under- and overspend. Thus, P_1 injects this diagnostic information into the decision-making process.

Optimal reallocation. To identify the candidate regions for reallocation, we compute 95% confidence interval around the optimal ad spends and resulting profit (using the distributions of the estimated model parameters). Figure 3 shows that the actual ad spends in Regions 1–5 lie within the 95% intervals, whereas those nationally and in Regions 6 and 7 exceed the confidence interval. Thus, the firm overspends in Regions 6 and 7 and at the national level. If managers reallocate budget to reduce ad spends in Regions 6 and 7 and nationally to fall within the confidence interval, profit increases between 1.7% and 24.1% (see Figure 3). By eliminating these misallocations but keeping the total budget unchanged, sales would increase by 1.01% and profit by 3.08%. This result reinforces the previous finding (e.g., Mantrala, Sinha, and Zoltners 1992) that a budget reallocation begets larger benefits than a change in overall budget. At the optimal allocation (i.e., setting the total

budget and its allocation in accordance with P_1), sales would decrease by approximately 12.14%, but profit would increase by 5.07%. This profit increase not only differs statistically from zero but also is larger than the profit the firm generates from its current actions.

In conclusion, we acknowledge that managers may forgo part of the profit to gain larger volumes—for example, to negotiate on price with retailers.[3] Indeed, overadvertising to increase volumes further supports the view that managers do not operate optimally. On the basis of these results, our discussions with the participating company influenced them to rethink their allocations of ad budget across national and regional advertising.

DISCUSSION

Time-Varying Parameters

Prior research has shown that model parameters for ad effectiveness and carryover effect may vary over time (e.g., Bass et al. 2007; Bronnenberg, Mahajan, and Vanhonacker 2000; Jedidi, Mela, and Gupta 1999; Mela, Gupta, and Lehmann 1997; Naik, Mantrala, and Sawyer 1998; Winer 1979).[4] However, the proposed model assumes constancy of parameters. Thus, we test this assumption in the Web Appendix (http://www.marketingpower.com/jmr_webappendix) using the Cusum test. As mentioned, we find that the confidence bounds contain the cumulative residuals, providing empirical support for parametric invariance.

Nonetheless, data from other markets might exhibit time-varying parameters. In addition to time, parameters may evolve as a result of multiple region-specific covariates $Z_i(t)$ (e.g., demographics, income, purchasing power). To accommodate such dynamic effects, let $(d\beta_i/dt) = p[Z_i(t), t]$, $(d\alpha_i/dt) = q[Z_i(t), t]$, $(d\delta_i/dt) = r[Z_i(t), t]$, and $(d\gamma_{ij}/dt) = u[Z_i(t), Z_j(t), t]$, where $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, and $u(\cdot)$ specify the process functions for parametric evolution (see, e.g., Gatignon and Hanssens 1987). Consequently, we augment the state space to include all $(K + 3)$ dynamics for each region: $[(dS_i/dt), (d\beta_i/dt), (d\alpha_i/dt), (d\delta_i/dt), (d\gamma_{ij}/dt)]'$. Then, we solve the multistate [dimension $K \times (K + 3)$] and multicon-

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[3]We thank an anonymous reviewer for this insight.
[4]We thank an anonymous reviewer for suggesting this extension.

Figure 3
RE-ALLOCATION OF AD SPENDS AND PROFIT
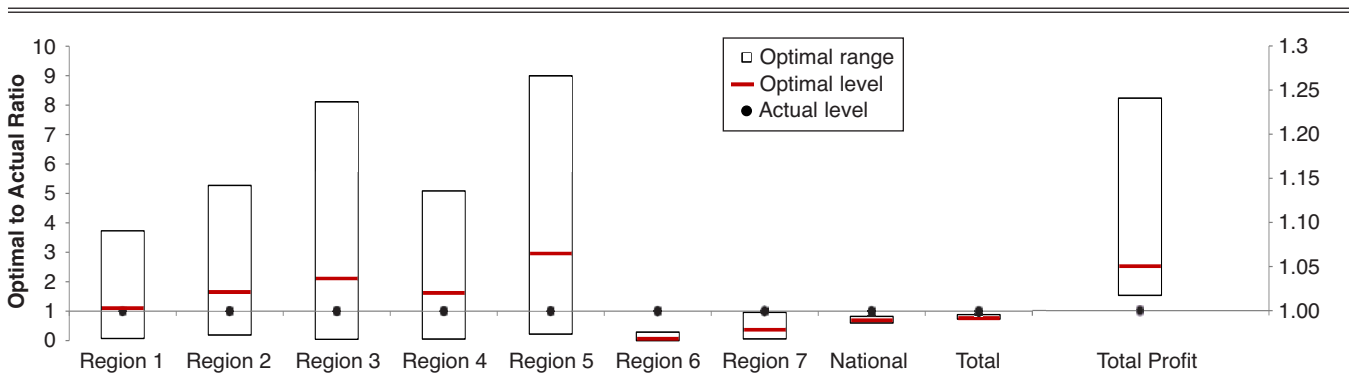
Notes: Actual level is normalized to unity.

trol (dimension K + 1) problem and generalize P_1 to markets with dynamically evolving parameters.

P_2: For time-varying parameters, the optimal regional and national advertising expenditures are as follows:

$$R_i^*(t) = \frac{1}{4}[\beta_i(t)b_i(t)]^2 \text{ for } i = 1, ..., K, \text{ and}$$

$$N^*(t) = \frac{1}{4}\left[\sum_{i=1}^{K} \alpha_i(t)b_i(t)\right]^2, \text{ where}$$

$$\begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_K(t) \end{bmatrix} = \begin{bmatrix} \rho + \delta_1(t) & -\gamma_{21}(t) & \cdots & -\gamma_{K1}(t) \\ -\gamma_{12}(t) & \rho + \delta_2(t) & \cdots & -\gamma_{K2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1K}(t) & -\gamma_{2K}(t) & \cdots & \rho + \delta_K(t) \end{bmatrix}^{-1} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_K \end{bmatrix}.$$

The total budget $B^*(t) = N^*(t) + \sum_{i=1}^{K} R_i^*(t)$ and its optimal

split $\phi^*(t) = N^*(t)/B^*(t)$.

(For a proof, see the Web Appendix at http://www.marketingpower.com/jmr_webappendix.)

A remarkable property of the optimal spends $[R_i^*(t), N^*(t)]$ is that they do not depend on the process functions $p(\cdot)$, $q(\cdot)$, $r(\cdot)$, $u(\cdot)$; they rely only on the resulting parameter values $[\alpha_i(t), \beta_i(t), \delta_i(t), \gamma_{ij}(t)]$. In other words, we learn that the optimal allocations depend on the outcome of *how much* the parameters changed rather than the process of *how* the change occurred.

Continuous- Versus Discrete-Time Models

We transform Equation 1 to continuous time for a reason.[5] In discrete-time models, the optimal decisions are solutions to stochastic difference equations, whereas in continuous-time models, we exploit the continuity of time to obtain nonstochastic (i.e., deterministic) differential equations. Because of the deterministic nature, the latter is more likely to yield analytical solutions than the former. We elaborate this issue in the Web Appendix (http://www.marketingpower.com/jmr_webappendix).

Effect of Discount Rate

Discount rate measures managers' impatience. As discount rate increases, managers become more impatient and present oriented. Thus, brand-building efforts shrink as managers become impatient. When we empirically compute the total spending for various discount rates from 0% to 12%, which spans the actual range stated in the company's annual reports (4.25% to 6.25%), we find that the total spending decreases by .56% for every 1% increase in the discount rate (see Figure 4). Finally, we observe that the actual spending exceeds the optimal spending under $\rho = 0$, reinforcing our findings that the firm overspends.

Pulsing Versus Even Spending

The optimal allocations this approach recommends yield even spending policies. Prior research has shown that puls-

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[5]We thank an anonymous reviewer for inquiring into the benefits of this approach.

Figure 4
EFFECT OF DISCOUNT RATES ON TOTAL BUDGET
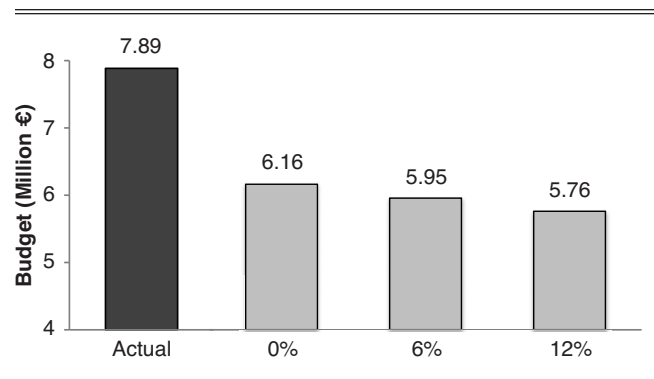
ing is near optimal when parameters vary over time (Naik, Mantrala, and Sawyer 1998) or the response function is S-shaped (e.g., Mahajan and Muller 1986). We conduct the Cusum test to rule out time-varying parameters (see the section "Robustness Checks") and compute information criteria to reject the S-shaped response function. Because the conditions required for pulsing do not apply to our setting, the prescribed even spending policy is optimal. For situations in which parameters vary over time, we derived P_2, which furnishes time-varying optimal allocations.

CONCLUSION

Recent marketing research (e.g., Albuquerque, Bronnenberg, and Corbett 2007; Ataman, Mela, and Van Heerde 2007; Bronnenberg, Dhar, and Dubé 2007a) has advanced the estimation of spatial or spatiotemporal models in marketing context, but important managerial questions have remained unanswered. Specifically, how much should companies spend on advertising, how much of it should be set aside for national versus regional advertising, and how should they allocate the regional dollars to support the different regions?

To provide systematic answers, we propose a spatiotemporal model of advertising that accounts for sales dynamics, neighborhood effects, spatial heterogeneity, and spatial and serial dependence. Because of spatial and serial dependence, a correlated multivariate Brownian motion drives the sales dynamics, which in turn results in a second-order differential equation for the value function with multiple states (i.e., regional sales) and multiple controls (i.e., regional advertising expenditure). We solve this normative problem analytically (see the Web Appendix at http://www.marketingpower.com/jmr_webappendix) to derive closed-form expressions for the optimal total budget and its optimal regional allocations for constant parameters (see P_1) and time-varying parameters (see P_2). In addition, we develop a method for estimation and inference of the proposed model, thereby extending Baltagi et al. (2007).

Our empirical analysis furnishes evidence for the presence of dynamic effects, neighborhood effects, spatial heterogeneity, and spatial and serial dependencies. Both the carryover effects and the effectiveness of advertising are significant across all regions. Carryover effects vary across regions by 11%, and regional and national advertising effectiveness vary by approximately 39.6% and 37.1%, respec-

tively. Neighborhood effects are positive and directly affect the optimal allocations. Spatial dependence is positive and indirectly affects the optimal allocations through the precision of parameter estimates. Serial dependence is negative, which reveals that oscillatory shocks emerge as a result of unobservable factors (e.g., stockpiling by consumers and retailers). All parameters are estimated efficiently (e.g., t-values range from 1.9 to 14.5). If significance is difficult to detect, shrinkage methods, such as the hierarchical Bayesian approach (Bass et al. 2007), direct constraints (Naik and Tsai 2005), or the Lasso (Tibshirani 1996), can be applied.

Our normative analysis shows that managers misallocate resources at both the national and regional levels. The total monthly budget should be reduced from €7.9 million to €5.9 million, and its split to national versus regional advertising should be changed from 92.4% to 85.9%. In addition, we observe specific regional misallocations; for example, Regions 6 and 7 overspend by 89.5% and 82.8%, respectively, and Regions 3 and 5 underspend by 115.9% and 191.5%, respectively. By reducing both overspending and misallocations, optimal reallocation would enhance profit by 5.07%. Thus, companies can make informed decisions by using the proposed method for optimal spatiotemporal allocation of advertising budgets across different regions.

We close by identifying an avenue for further research. Because ad spending is committed by contract with media companies several months in advance through the up-front market (see Belch and Belch 2004, p. 358; Raman and Naik 2004, p. 11; Tellis 1998, p. 351), firms cannot change their media schedules in response to competition in the short run. Likewise, competitors also are committed and lack the flexibility to change media plans in response to the focal firm's advertising. Accordingly, academic literature shows that competitors seldom respond (e.g., Steenkamp et al. 2005). Thus, competitive response is minimal in the short run. However, this phenomenon may differ over a longer time horizon, and so we encourage further research to understand the role of competition.

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