

Planning Marketing-Mix Strategies in the Presence of Interaction Effects

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Companies spend millions of dollars on advertising to boost a brand's image and simultaneously spend millions of dollars on promotion that many believe calls attention to price and erodes brand equity. We believe this paradoxical situation exists because both advertising and promotion are necessary to compete effectively in dynamic markets. Consequently, brand managers need to account for interactions between marketing activities and interactions among competing brands. By recognizing interaction effects between activities, managers can consider interactivity trade-offs in planning the marketing-mix strategies. On the other hand, by recognizing interactions with competitors, managers can incorporate strategic foresight in their planning, which requires them to look forward and reason backward in making optimal decisions. Looking forward means that each brand manager anticipates how other competing brands are likely to make future decisions, and then by reasoning backward deduces one's own optimal decisions in response to the best decisions to be made by all other brands. The joint consideration of interaction effects and strategic foresight in planning marketing-mix strategies is a challenging and unsolved marketing problem, which motivates this paper.

This paper investigates the problem of planning marketing mix in dynamic competitive markets. We extend the Lanchester model by incorporating interaction effects, constructing the marketing-mix algorithm that yields marketing-mix plans with strategic foresight, and developing the continuous-discrete estimation method to calibrate dynamic models of oligopoly using market data. Both the marketing-mix algorithm and the estimation method are general, so they can be applied to any other alternative model specifications for dynamic oligopoly markets. Thus, this dual methodology augments the decision-making toolkit of managers, empowering them to tackle realistic marketing problems in dynamic oligopoly markets.

We illustrate the application of this dual methodology by studying the dynamic Lanchester competition across five brands in the detergents market, where each brand uses advertising and promotion to influence its own market share and the shares of competing brands. Empirically, we find that advertising and promotion not only affect the brand shares (own and competitors') but also exert interaction effects, i.e., each activity amplifies or attenuates the effectiveness of the other activity. Normatively, we find that large brands underadvertise and overspend on promotion, while small brands underadvertise and underpromote. Finally, comparative statics reveal managerial insights into how a specific brand should respond optimally to the changes in a competing brand's situation; more generally, we find evidence that competitive responsiveness is asymmetric.

Key words: continuous-discrete estimation; dynamic competition; interaction effects; marketing-mix planning; strategic foresight; two-point boundary value problem

History: This paper was received June 19, 2001, and was with the authors 14 months for 4 revisions; processed by William Boulding.

1. Introduction

American corporations collectively spend over \$500 billion on marketing activities; even individual companies such as Procter and Gamble spend several billion dollars on advertising and promotion. Consequently, the optimal allocation of marketing resources to multiple activities—referred to as “planning the marketing mix”—is of paramount importance (see

Manrala 2002 for literature review). In the extant literature, dynamic planning models such as Naik et al. (1998) and Silva-Risso et al. (1999) provide decision-support tools to determine advertising schedules and promotional calendars, respectively. These decision-support models, however, ignore the game-theoretic principle of strategic foresight, a notion that requires the brand manager to *look forward*, i.e., anticipate how other competing brands are likely to make future

decisions, and then *reason backward*, i.e., deduce one's own optimal decisions in response to the best decisions to be made by all other brands. On the other hand, dynamic game-theoretic models that advocate strategic foresight ignore the role of interactions among multiple marketing activities. Such interactions are central to the marketing-mix concept, which "...emphasizes that marketing efforts create sales synergistically rather than independently" (see Gatignon and Hanssens 1987, p. 247; Lilien et al. 1992, p. 5; also see Gatignon 1993 for a literature review).

The joint consideration of both strategic foresight and interaction effects in dynamic response models represents an important gap in the marketing literature. For example, Fruchter and Kalish (1998, p. 22) acknowledge

"...the limitations of current studies [not] to take into account the interactions among the different instruments. A challenge which we see for a future direction is to develop a model which incorporates interactions between promotional instruments."

The challenging problems arise for the following two reasons. First, as we show later, in the presence of interaction effects, the optimal plans for all activities are interdependent, thereby requiring managers to account for the interactivity trade-offs in budget allocations. In other words, the optimal level to spend on advertising depends on the optimal level to spend on promotion (and vice versa). Second, managers need to know the joint effectiveness of marketing activities of all other brands to be able to determine their own optimal marketing-mix plans. This demands a new methodology for estimating dynamic models of oligopoly markets using market data. Thus, both the substantive problems—the determination of optimal marketing-mix strategies and the estimation of dynamic models for oligopoly markets—are unsolved research topics because the necessary methodology does not yet exist in marketing, economics, or management science (see Erickson 1991, Kamien and Schwartz 1991, Dockner et al. 2000).

Given this gap in the literature, one cannot answer basic questions of managerial interest: Do advertising and promotion amplify or attenuate their impact on market outcomes (e.g., brand share) when used together? How should managers allocate resources to advertising and promotion in the presence of interaction effects? What is the level of optimal budget and its allocation to promotional activities in the presence of strategic foresight? Is own (or competitor's) brand underadvertising or overpromoting, or both? If brand A's interaction effect increases, should brand B optimally respond by increasing advertising or increasing promotion?

To help answer such questions, we develop two methods: (i) a marketing-mix algorithm to plan optimal marketing-mix strategies and (ii) an estimation method to determine the effectiveness of marketing activities and their interaction effects for each brand in dynamic competitive markets.

The proposed marketing-mix algorithm solves the multiple-player differential game resulting from the dynamic models of oligopoly markets. Specifically, it yields optimal marketing-mix strategies that are (a) in equilibrium across multiple brands and over time, (b) accounts for intertemporal trade-offs across multiple periods (i.e., now versus later), and (c) balances interactivity trade-off among multiple marketing activities (e.g., advertising versus promotion). Because this algorithm solves a general nonlinear two-point boundary value problem, its applicability extends to several differential game models in marketing, not only the Lanchester model that we present for the sake of exposition (see Remark 2 for details). In addition, it furnishes both the open-loop and closed-loop marketing-mix strategies (see the unabridged manuscript, which is available from the authors upon request).

The proposed estimation method calibrates a simultaneous system of coupled differential equations that arise in the differential game competition in an oligopoly. We note that standard time-series techniques are applicable for discrete-time difference equations (e.g., ARIMA-type models; see Blattberg and Neslin 1990, Ch. 9) but not for continuous-time models based on differential equations. Hence, we develop an estimation approach for estimating continuous-time models using discrete-time observed data (e.g., weekly). Specifically, this approach provides maximum-likelihood estimates and standard errors that incorporate (a) the temporal dependence in brand shares, (b) their nonstationary dynamics, and (c) the marketing-mix interactions and competitive interdependencies.

To illustrate the application of these dual methodologies, we extend the Lanchester model in §2. Section 3 analyzes the N -player differential game and constructs the marketing-mix algorithm. Section 4 develops the continuous-discrete estimation approach to estimate dynamic oligopoly models. We present an illustrative example in §5 and conclude in §6.

2. Extended Lanchester Model

We extend the classical Lanchester model by introducing the following phenomena: multiple brands in competition, multiple marketing activities that each brand employs to influence its own and competing brands' shares, and interactions between marketing activities.

Table 1 Related Literature

	Single activity	Multiple activities with no interactions	Multiple activities with interactions
Monopoly	Sethi (1977) Little (1979)	Naik and Raman (2003)	Naik and Raman (2003)
Duopoly	Erickson (1991)	Chintagunta and Vilcassim (1994)	This study
Oligopoly	Dockner et al. (2000)	Fruchter and Kalish (1998)	This study

2.1. Multiple Brands

In the classical Lanchester model, two brands compete for market shares using advertising. Let $m_i(t)$ and $u_i(t)$ denote market share and advertising effort, respectively, for brand i , $i = 1$ and 2. Then, the Lanchester model is

$$\dot{m}_1 = \alpha_1 u_1 (1 - m_1) - \alpha_2 u_2 m_1, \quad (1)$$

where $\dot{m}_i = dm_i/dt$ is the rate of change in m_i , and α_i denotes advertising effectiveness. In Equation (1), the change in market share of the first brand is due to two sources. First, the first brand captures some market share from the other brand $m_2 = (1 - m_1)$ via its own advertising (i.e., by increasing α_1 or u_1). Second, it loses market share when the other brand increases advertising or improves effectiveness (i.e., as u_2 or α_2 increases).

Next, we extend this model to an oligopoly market with N brands. For each brand i , $i = 1, \dots, N$, let f_i be the force of one's own marketing activities; for example, $f_1 = \alpha_1 u_1$ and $f_2 = \alpha_2 u_2$ in (1). Then, generalizing (1) to N brands, the market share of brand 1 evolves as follows:

$$\begin{aligned} \frac{dm_1}{dt} &= f_1(1 - m_1) - f_2 m_1 - f_3 m_1 \cdots \\ &\quad - f_N m_1. \end{aligned} \quad (2)$$

We further simplify (2) to obtain $\dot{m}_1 = f_1 - Fm_1$, where $F = \sum_{i=1}^N f_i$ denotes the marketing force of all the brands (including the first brand). Thus, we formulate the N -brand Lanchester model as the simultaneous system of coupled differential equations:

$$\begin{bmatrix} \dot{m}_1 \\ \vdots \\ \dot{m}_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} - \begin{bmatrix} F & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_N \end{bmatrix}. \quad (3)$$

2.2. Multiple Marketing Activities with Interaction Effects

Marketing activities such as advertising and promotion generate the marketing force of a brand. Denoting advertising and promotional efforts by (u_i, v_i) , we specify the marketing force of brand i , $i = 1, \dots, N$,

$$f_i = \alpha_i u_i + \beta_i v_i + \gamma_i u_i v_i, \quad (4)$$

where (α_i, β_i) represent the advertising and promotion effectiveness, respectively, and γ_i is the

interaction effect. Equation (4) states that the marketing force depends not only on the weight of advertising and promotion but also on their interaction; both the weights and effectiveness are brand-specific. Furthermore, each activity can amplify or attenuate the effectiveness of other activities. Assuming positive effect of advertising, when brand i increases u_i , it not only increases its brand share m_i (directly via Equation (3)) but also influences the effectiveness of the other activity v_i because of the nonzero γ_i (indirectly via Equation (4)). The two activities strengthen the marketing force by creating synergy when $\gamma > 0$; they weaken the marketing force when $\gamma < 0$. Finally, we focus on two marketing activities because no new conceptual issues arise in extending (4) to multiple activities.

REMARK 1. Table 1 shows how the extended Lanchester model augments the marketing literature on important dimensions. Specifically, Sethi (1977) and Little (1979) review the literature on dynamic monopoly models with single activity; Erickson (1991) and Dockner et al. (2000) summarize dynamic competitive models with a single activity. By introducing multiple activities, Chintagunta and Vilcassim (1994) and Fruchter and Kalish (1998) extend these differential game models but ignore interactions among these activities. By introducing interaction effects, which are central to the marketing-mix concept (see, e.g., Gatignon and Hanssens 1987; Gatignon 1993; Lilien et al. 1992, p. 5), Naik and Raman (2003) establish empirically the existence of synergy and offer theoretical insights into budgeting and allocation decisions. However, they investigate monopoly markets, which ignore a competitor's response to one's own budgeting and allocation decisions. Hence, the extended Lanchester model via (3) and (4) fills this gap in the literature by incorporating both the interaction effects and dynamic competition. Next, we analyze the differential game for this extended model and construct an algorithm that yields equilibrium marketing-mix plans.

3. Marketing-Mix Algorithm

3.1. Differential Game Formulation

Each brand i develops its advertising plan $u_i(t)$ and promotion plan $v_i(t)$ by maximizing its performance

index (Erickson 1991, p. 18):

$$J(u_i, v_i) = \int_{t=0}^T e^{-\rho t} \underbrace{\left[(p_i(t) - v_i(t)) m_i(t) - c(u_i(t)) \right]}_{\text{margin share advertising cost}} dt, \quad (5)$$

$\underbrace{\text{net value}}_{\text{discounted net value}}$

which quantifies the performance of a strategy pair $(u_i(t), v_i(t))$. In (5), ρ denotes the discount rate, p_i is the price of brand i ,¹ v_i is the size of the deal, m_i is the market share, u_i is the advertising effort, and $c(u)$ is the cost of expending u . The associated Hamiltonian is

$$H^i(u_i, v_i) = (p_i - v_i)m_i - c(u_i) + \lambda_i(f_i - Fm_i), \\ i = 1, \dots, N, \quad (6)$$

where $\lambda_i(t)$ is the co-state variable for brand i . Below we assume the cost function $c(u) = bu + u^2/2$ (e.g., Fruchter and Kalish 1998), although the proposed marketing-mix algorithm permits other functional forms (see Remark 2 later).

3.2. Equilibrium Strategies

3.2.1. Optimal Advertising. By differentiating (6) with respect to u_i , and solving $\partial H^i / \partial u_i = 0$, we obtain

$$u_i^* = \lambda_i(\alpha_i + \gamma_i v_i)(1 - m_i) - b, \quad i = 1, \dots, N. \quad (7)$$

Equation (7) specifies the optimal advertising strategy for each brand i , which depends on one's own market share (m_i), ad effectiveness (α_i), the cost parameter b , and other brands' strategies via the co-state variable $\lambda_i(t)$, whose dynamics we derive in Equation (12). It further reveals that, in the absence of interaction effects (i.e., when $\gamma_i = 0$), advertising plans do not directly depend on the promotion plans. In contrast, in the presence of interaction effects (i.e., when $\gamma_i \neq 0$), optimal advertising $u_i^*(t)$ depends directly on the promotion plan $v_i(t)$. Hence, the *interaction effect qualitatively changes the planning of the marketing-mix by requiring managers to account for interactivity trade-offs*.

3.2.2. Optimal Promotion. By differentiating (6) with respect to v_i , we obtain

$$\partial H^i / \partial v_i = -m_i + \lambda_i(1 - m_i)(\beta_i + \gamma_i u_i), \quad (8)$$

¹ In practice, managers should use the net price $\tilde{p}_i(t) = p_i(t) - c_i(t)$ to account for the unit cost of goods sold, c_i . In developing the marketing-mix algorithm, however, we assume it zero to minimize notation without loss of generality. When using the algorithm, managers can include the unit cost and perform the sensitivity analysis (as we do in the empirical example).

which will not yield the optimal promotion plan $v^*(t)$ upon equating to zero. Hence, we define the switching function,

$$D_i = -m_i + \lambda_i(1 - m_i)(\beta_i + \gamma_i u_i), \quad (9)$$

which yields the optimal bang-bang policy,²

$$v_i = \begin{cases} \bar{v}_i & D_i > 0 \\ 0 & D_i < 0 \end{cases}, \quad i = 1, \dots, N, \quad (10)$$

where \bar{v}_i denotes the maximum deal amount for brand i at time t . The promotion plan is either "on" (i.e., offer price discount) or "off," so we express it by $v_i(t) = \bar{v}_i I(D_i > 0)$, where $I(\cdot)$ denotes an indicator variable 1 or 0 depending on whether the switch D_i is positive or not. This switch depends on the co-state variable $\lambda_i(t)$, which involves the strategies of all competing brands. Consequently, each brand forms an expectation of whether other brands would promote or not. By taking expectations, we find that brand i 's policy is given by

$$\begin{aligned} v_i^* &= \bar{v}_i E[I(D_i > 0)] \\ &= \bar{v}_i \Pr(-m_i + \lambda_i(1 - m_i)(\beta_i + \gamma_i u_i) > 0) \\ &= \bar{v}_i \Pr\left(m_i < \frac{\lambda_i(\beta_i + \gamma_i u_i)}{1 + \lambda_i(\beta_i + \gamma_i u_i)}\right) \\ &= \frac{\bar{v}_i \lambda_i(\beta_i + \gamma_i u_i)}{1 + \lambda_i(\beta_i + \gamma_i u_i)}, \end{aligned} \quad (11)$$

where the last equality follows by noting that the support of brand share m_i is the unit interval $(0, 1)$, which like the standard uniform variable U yields $\Pr(U < c) = c$. The fixed point of (7) and (11) yields the equilibrium pair (u^*, v^*) as a function of the co-state variables $\lambda(t)$.

3.2.3. Co-State Dynamics. We derive the co-state dynamics by evaluating $\dot{\lambda}_i = \rho \lambda_i - \partial H^i / \partial m_i$ and obtain

$$\begin{bmatrix} \dot{\lambda}_1 \\ \vdots \\ \dot{\lambda}_N \end{bmatrix} = \begin{bmatrix} \rho + F & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \rho + F \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} - \begin{bmatrix} p_1 - v_1 \\ \vdots \\ p_N - v_N \end{bmatrix}, \quad (12)$$

where $\dot{\lambda}_i = d\lambda_i/dt$, and $i = 1, \dots, N$. The co-state dynamics incorporate the fundamental principle of

² We derive the limiting condition when the bang-bang policy could degenerate to a singular value (i.e., constant discount). The condition requires that every brand satisfies the equality: $(1 - m_1) \cdot (p_1 - v_1)(\beta_1 + \gamma_1 u_1)/m_1 = \cdots = (1 - m_i)(p_i - v_i)(\beta_i + \gamma_i u_i)/m_i = \cdots = (1 - m_N)(p_N - v_N)(\beta_N + \gamma_N u_N)/m_N$. If this equality holds in some market, every brand would offer a fixed discount, thus lowering its regular price effectively, and the resulting marketing-mix problem reduces to one of finding the optimal advertising plans, which is a special case of the above problem.

strategic foresight, which requires managers to look forward (i.e., forecast outcomes and anticipate competitive moves) and reason backward (i.e., deduce the best strategy via backward induction). Specifically, to look forward, we apply the state dynamics in (3) and (4); to anticipate competitive moves, we evaluate the best strategies using (7) and (11); to reason backward, we solve the co-state dynamics in (12) from the terminal period to the initial time. Thus, we can systematically create equilibrium marketing-mix plans with strategic foresight by stacking the two systems of differential equations (3) and (12) and solving them simultaneously together with (4), (7), and (11). To solve the resulting two-point boundary value problem (TPBVP), we next construct a marketing-mix algorithm.

3.3. Marketing-Mix Algorithm

We note that extant algorithms in marketing (e.g., Deal 1979, Thompson and Teng 1984, Erickson 1991) are not applicable to solve the above TPBVP because they ignore the coupling between and within the state and the co-state equations. For example, the coupling of differential equations arises because of the market force $F = f_i + \sum f_j$ in both (3) and (12). Hence, we construct the following algorithm that not only accounts for interequation coupling but also permits a general nonlinear TPBVP specified by

$$\dot{y} = g(y; \Theta), \quad (13)$$

where y and \dot{y} are vectors with $2N$ elements (i.e., N state and N co-state variables), and Θ denotes the parameter space. Specifically, the algorithm consists of the four steps:

Step 1. Boundary conditions. Choose M grid points on a timeline, $k = 1, 2, \dots, M$, and specify two points for the boundary conditions: the initial $k = 0$, and the terminal $k = M + 1$.

Step 2. Finite difference equations. Create difference equations $E_k \equiv y_k - y_{k-1} - hg(w_k; \theta)$, where $h = t_k - t_{k-1}$, and $w_k = (y_k + y_{k-1})/2$. Then stack all the equations E_1, E_2, \dots, E_{M+1} one below the other, and denote this stack by the vector-valued function $G(x)$, where $x = \text{vec}(y_1, y_2, \dots, y_M)$.

Step 3. Nonlinear root-finding. Solve the equations $G(x) = \mathbf{0}$ via Newton-Rahpson method.

Step 4. Optimal marketing-mix plans. Using state and co-state trajectories obtained from step 3, determine the marketing-mix plans via (7) through (10), and evaluate their performance via (5).

For further details, see Table 3 in the unabridged manuscript (available from the authors).

REMARK 2. This algorithm is not restricted to the extended Lanchester model; it encompasses other alternative model specifications for dynamic oligopoly markets. Its generality stems from the mathematical structure of a TPBVP, which subsumes several

differential games in marketing because all of them are expressible in the form of Equation (13). After expressing a game as Equation (13), the above algorithm yields the optimal strategies—only the shape of the link function $g(\cdot)$ and the set of input arguments (y, Θ) change to accommodate different specifications and phenomena. Specifically, it accommodates different state dynamics instead of (3); different marketing forces instead of (4); different marketing objectives instead of (5); and different phenomena such as channel members, reference prices, or cost functions. Furthermore, it is not restricted to open-loop marketing-mix plans; it also furnishes closed-loop marketing-mix strategies (see §5.3 in the unabridged manuscript). Next, we develop an approach to quantify the effectiveness and interaction effects for each brand in the oligopoly and to infer their statistical significance based on market data.

4. Continuous-Discrete Estimation Method

We note that the brand-share dynamics in (3) are in continuous time; that is, the time parameter t in the model is continuously differentiable on any interval (\underline{t}, \bar{t}) . In contrast, brand-share data arrive at discrete points in time (e.g., weeks). In other words, the time parameter t in the data series is not continuously differentiable; rather it takes discrete values in the integer set $\{1, 2, 3, \dots, T\}$. To resolve this mismatch, we develop an approach to estimate continuous-time models using discrete-time data.

Let m_k denote the $N \times 1$ vector of brand shares at time t_k , where k is an integer in the index set $\{1, 2, \dots, T\}$. Using (3), we integrate over the interval $(t_{k-1}, t_k]$ to obtain the exact equation relating the brand shares at two discrete time points:

$$m_k - m_{k-1} = \int_{t_{k-1}}^{t_k} dm = \int_{t_{k-1}}^{t_k} \left(\frac{dm}{dt} \right) dt = \int_{t_{k-1}}^{t_k} (\dot{m}) dt,$$

which, upon simplification, yields the matrix transition equation

$$m_k = \Phi_k m_{k-1} + \delta_k, \quad (14)$$

where $\Phi_k = \exp(A_k)$ is a $N \times N$ transition matrix, $\exp(\cdot)$ is the matrix exponentiation function, A_k is a diagonal $N \times N$ matrix with $(-F_k)$ as the elements on the principal diagonal, and the $N \times 1$ drift vector $\delta_k = (\exp(A_k) - I_N) A_k^{-1} f_k$, where I_N is an N -dimensional identity matrix, and $f_k = (f_{1k}, \dots, f_{Nk})'$ is the vector of marketing forces for the N brands.

Let y_{ik} denote the observed share for brand i at time t_k . Because brand shares sum to 100%, once we know $(N - 1)$ brand shares, the realized share for the remaining brand, y_{Nk} , provides no new information, because it must equal $100 - \sum_{i=1}^{N-1} y_{ik}$. Consequently,

we need information on any $(N - 1)$ brands, which we link to (14) via the observation equation

$$\begin{bmatrix} y_{1k} \\ \vdots \\ y_{N-1,k} \end{bmatrix} = \begin{bmatrix} 100 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 100 \end{bmatrix} \begin{bmatrix} m_{1k} \\ \vdots \\ m_{N-1,k} \end{bmatrix}, \quad (15)$$

where $Y_k = (y_{1k}, \dots, y_{N-1,k})'$ are the observed brand shares (in percentages).

We express (14) and (15), together, in the state-space form (Shumway and Stoffer 2000):

$$\begin{aligned} Y_k &= z m_k + \varepsilon_k, \\ m_k &= \Phi_k m_{k-1} + \delta_k + \nu_k, \end{aligned} \quad (16)$$

where z is a diagonal matrix with its principal elements equal to 100, and the error-terms follow the normal distributions, $\varepsilon_k \sim N(0, H)$ and $\nu_k \sim N(0, Q)$, where H is the diagonal matrix $\sigma^2 I_{N-1}$ and Q is the diagonal matrix, $\text{diag}(\sigma_1^2, \dots, \sigma_{N-1}^2)$. We then compute the likelihood function for observing the trajectory of market shares of all the brands in an oligopoly by

$$\begin{aligned} L(\Theta; Y) &= h(Y_1, Y_2, \dots, Y_T) \\ &= h(Y_1) \times h(Y_2 | Y_1) \times h(Y_3 | (Y_1, Y_2)) \\ &\quad \times \cdots \times h(Y_T | (Y_1, Y_2, \dots, Y_{T-1})) \\ &= \prod_{k=1}^T h(Y_k | \mathfrak{I}_{k-1}), \end{aligned} \quad (17)$$

where $h(Y_1, \dots, Y_T)$ denotes the joint density function, $h(Y_k | \mathfrak{I}_{k-1})$ represents the conditional density function, whose moments are obtained from the Kalman filter recursions, and the information set $\mathfrak{I}_k = \{Y_1, Y_2, \dots, Y_k\}$ contains the market history up to time t_k . By maximizing (17), we obtain the maximum-likelihood estimates

$$\hat{\theta} = \underset{\theta \in \Theta}{\text{Arg Max}} \ln(L(\Theta; Y)). \quad (18)$$

For statistical inference, we obtain the standard errors by taking the square-root of the diagonal elements of the covariance matrix:

$$\text{Var}(\hat{\theta}) = \left[-\frac{\partial^2 \ln(L(\Theta; Y))}{\partial \Theta \partial \Theta'} \right]_{\theta=\hat{\theta}}^{-1}. \quad (19)$$

REMARK 3. The estimation of continuous-time models (e.g., Equation (3)) via the ordinary least-squares (OLS) regression results in biased estimates for the following three reasons. First, the OLS regression requires independence between brand shares at two points in time, Y_k and $Y_{k'}$. Clearly, such temporal independence cannot exist because the differential equation (3) explicitly induces dependence over time. Second, this temporal dependence does not

vanish or diminish because brand-share dynamics are nonstationary, a fact evident from the transition matrix Φ_k in (14), which is time-varying because it depends on k . Finally, the shares of all N brands in an oligopoly are highly interdependent on each other's marketing activities. Specifically, we observe from (14) that both Φ_k and δ_k depend nonlinearly on the marketing forces F and f_i . Therefore, in the proposed method, we explicitly account for temporal dependence, nonstationary dynamics, promotional mix interactions, and competitive interdependencies in estimating dynamic oligopoly models. We next illustrate its application to the dynamic oligopoly of detergents brands.

5. Empirical Application

5.1. Data

We use single-source data for the five detergents brands Tide, Wisk, Era, Solo, and Bold, whose average percentage shares are 43, 19, 16, 13, and 9, respectively. The data consist of the purchase histories, prices paid, cents-off deal, and TV viewing by households over 84 weeks (see Winer 1993 for details). Because the Lanchester model is specified at the brand level, researchers usually use market- or store-level data for calibration (e.g., Little 1979, Chintagunta and Vilcassim 1994). While aggregate market data furnish information on competing brands for many nongrocery products, the household-level data provide opportunities not available in aggregate data. For example, we can identify the advertisements seen by each household, the deal it received, and the brands it bought subsequently. Using this detailed information, we can operationalize better measures of advertising and promotion. For example, to measure advertising, we use opportunity-to-see (OTS), which is defined as the number of exposures viewed by a household within a purchase cycle (Lilien et al. 1992, pp. 272–274). For further details, see the unabridged version.

5.2. Empirical Results

5.2.1. Model Selection, Cross-Validation, and Hausman Test. We apply the proposed continuous-discrete estimation method to estimate several alternative model specifications: dynamic model with interactions, dynamic model without interactions, price effects with interactions, price effects without interactions, diminishing returns (square-root), increasing returns (quadratic). Based on information criteria (not presented here for brevity), we retain the extended Lanchester model in (3) and (4)—it is not only parsimonious but also fits the data well for all five brands. In addition, to assess predictive validity, we conduct cross-validation using

Table 2 Continuous-Discrete Kalman Filter Estimates

Parameters	Tide	Wisk	Era	Solo	Bold
Ad effectiveness, α_i	0.35 (0.07)	0.33 (0.11)	0.13 (0.04)	0.10 (0.03)	0.35 (0.26)
Promotion effectiveness, β_i	0.38 (0.09)	0.14 (0.04)	0.18 (0.05)	0.18 (0.04)	0.22 (0.12)
Ad-promotion interaction, γ_i	-0.33 (-0.08)	-0.13 (-0.22)	0.02 (0.06)	-0.08 (-0.06)	-0.35 (-0.62)
Transition noise, $\exp(b_i)$, b_i	-4.70 (-0.19)	-5.21 (-0.24)	-5.41 (-0.28)	-6.30 (-0.38)	—
Observation noise, σ			2.96 (1.07)		
Maximized log-likelihood, LL*			-852.55		

72 observations for calibration and 12 observations for out-of-sample validation, and we find high R^2 for out-of-sample forecasts across all five brands. Finally, we apply the Hausman test, find $m = 10.23$ and critical $\chi^2_{20, 0.95} = 31.41$, and thus cannot reject the null hypothesis of exogeneity of advertising and promotion. We present the parameter estimates and standard errors (in parentheses) in Table 2.

5.2.2. Advertising and Promotion Effectiveness. Table 2 indicates that ad and promotion effectiveness vary considerably in this detergents market. Specifically, ad effectiveness of all brands, except Bold, is significant (p -values < 0.05). Tide's advertising is the most effective, whereas Solo's advertising is the least effective. Similarly, promotion effectiveness of all brands, except Bold, is significant (p -values < 0.05). Tide's promotion is the most effective, while Wisk's promotion is the least effective.

5.2.3. Interaction Effects. Table 2 also shows that all brands, except Era, exhibit negative interactions between advertising and promotion. For assessing joint significance, we test the null $H_0: \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0$ by using the likelihood ratio (LR) test. The LR statistic is 20.6, which exceeds the critical value $\chi^2_5 = 11.1$ (p -value = 0.00096). Hence, we reject the null hypothesis of no interaction effects. In addition, multiple information criteria (not reported here for brevity) reinforce the finding that interaction effects exist in this detergents market. Thus, when advertising and

promotion are used together, their impact on brand shares is attenuated.

One interpretation of negative interaction, as in Mela et al. (1997) and Jedidi et al. (1999), is that "promotions are bad"—but not because they directly hurt brand shares. Indeed, the direct effect of promotion is positive, making it an effective marketing activity. Rather, promotion negatively moderates the impact of advertising, thereby reducing the effectiveness of advertising in building brands. Another interpretation of this negative interaction is that advertising lowers consumer sensitivity to promotion activities.

5.2.4. Optimal Advertising and Promotion. We apply the proposed marketing-mix algorithm to find optimal advertising and promotion and present the results in Table 3. We observe that all brands underadvertise (i.e., actual advertising is below the optimal level). In contrast, some brands overpromote whereas others underpromote. These findings are not sensitive to marginal changes in unit cost (not reported here for space constraints). Specifically, Tide and Wisk, who underadvertise, tend to promote excessively. It appears that large brands allocate fewer resources to advertising relative to promotion, a finding consistent with the notion of escalation of promotion due to managers' lack of strategic foresight (Leeflang and Wittink 2001, p. 120). On the other hand, some small brands (e.g., Era) both underadvertise and underpromote, perhaps due to lack of resources.

Table 3 Actual Versus Optimal Advertising and Promotion

Brands	Advertising (weekly opportunity-to-see in minutes)			Promotion (average deal in cents)		
	Actual	Optimal		Actual	Optimal	
Tide	18.0	25.6	Underadvertise	62	0	Overpromote
Wisk	6.2	24.9	Underadvertise	44	0	Overpromote
Era	9.9	20.6	Underadvertise	37	47	Underpromote
Solo	8.8	20.6	Underadvertise	39	47	Underpromote
Bold	3.3	18.6	Underadvertise	23	2	Overpromote

5.2.5. Optimal Competitive Responsiveness. Managers can gain further insights via comparative statics analyses. For example, if Tide's interaction effect increases, would Wisk respond with increased advertising or should it increase promotion? Would Solo's response differ from Tide's? Managers can answer these strategic questions by first applying the proposed marketing-mix algorithm to characterize the optimal advertising and promotion strategies for own and competitors brands, and then changing the parameter of interest (say, Tide's interaction effect), while keeping other parameters constant, to find the new optimal strategies.

Table 4 displays the comparative statics results with respect to interaction effects γ_i for each brand ($i = 1, \dots, 5$). Specifically, it reveals that the optimal competitive response is not symmetric. For example, based on the second column in Table 4, when Tide's negative interaction increases, it decreases advertising and increases promotion; all other brands' follow Tide's actions. By contrast, based on the third column in Table 4, when Wisk's negative interaction increases, it *increases* advertising and *decreases* promotion (unlike Tide's actions in column 2); other brands' follow suit, but not Tide whose best response is to decrease advertising and increase promotion. While this finding of asymmetry is new, the main point of this illustration is the ability afforded by the proposed marketing-mix algorithm for managers to gain such insights into their product markets. Thus, in practice, managers can also learn competitive responsiveness with respect to other parameters (e.g., what if ad or promotion effectiveness changes?).

REMARK 4. We close by noting a useful property of this marketing-mix algorithm. Consider a typical brand manager who decides whether to promote or not in each week of the annual horizon. This brand manager then encounters $2 \times 2 \times 2 \times \dots \times 52$ times $= 2^{52}$ possibilities, which represents *thousands of trillions of promotion plans*. Brand managers cannot enumerate trillions of plans to select the best one, even with the availability of modern computers. Consequently, the determination of the "best" promotion schedule is practically impossible. Hence, it is remarkable that the proposed marketing-mix algorithm finds the optimal plan in a few minutes via the powerful concept

of "switching functions" (see Equation (9) and the unabridged manuscript for details).

6. Discussion

We acknowledge limitations of the model and identify avenues for future research.

6.1. Alternative Specifications

Wittink et al. (1988, p. 4) make the point forcefully that, although managers have strong interest in market share, category sales may fluctuate over time. Market share models, such as the Lanchester model, ignore the fluctuations in category sales. To capture this feature, let $M(t)$ denote category volume at time t so that Equation (3) becomes $\dot{S}_i = M(t)f_i(t) - F(t)S_i(t)$, where $S_i(t)$ is the sales of brand i ($i = 1, \dots, N$). Then, the first-order conditions for the resulting sales game are structurally equivalent to Equations (7) and (9). Consequently, we can apply the proposed marketing-mix algorithm to obtain the optimal strategies in the presence of fluctuating category sales (see Remark 2). Thus, managers can investigate how over- or under-spending changes across various patterns of category sales (e.g., seasonality).

Another limitation of the Lanchester model is that it misses lead and lag effects for advertising and promotion. Recently, van Heerde et al. (2000) studied the role of lead and lag effects of promotion and detected pre- and postpromotion dips in store-level data. To incorporate this feature in the Lanchester model, let $u_{t+\tau}^i$ and $v_{t+\tau}^i$ denote the lead variables and $u_{t-\tau}^i$ and $v_{t-\tau}^i$ be the lagged variables for advertising and promotion, respectively. Then, by extending (4), we obtain the marketing force $f_i = \alpha_i u_i + \beta_i v_i + \gamma_i u_i v_i + \sum_{\tau=1}^{K^*} \varpi_{i,\tau} u_{i,t+\tau} + \sum_{\tau=1}^K \omega_{i,\tau} u_{i,t-\tau}$, for each brand $i = 1, \dots, N$, where K^* and K denote the number of lead and lagged variables in the model. Note that these new regressors in the extended specification naturally (i.e., without any modifications) enter into the transition matrix Φ_k and the drift vector δ_k via F_k and f_k in Equation (14). Consequently, managers can apply the proposed estimation method (see §4) to estimate the lead effects $(\hat{\varpi}_{i,1}, \hat{\varpi}_{i,2}, \dots, \hat{\varpi}_{i,K^*})$ as well as the lagged effects $(\hat{\omega}_{i,1}, \hat{\omega}_{i,2}, \dots, \hat{\omega}_{i,K})$. To determine the optimal values of K^* and K , classical information criteria (e.g., AIC, BIC) can be applied for parametric model specifications. When model specification is semiparametric or nonparametric (for example, local polynomial regression; see van Heerde et al. 2004), then the corrected criterion AIC_C (Naik and Tsai 2001) or RIC (Naik and Tsai 2003) needs to be applied. Thus, managers can investigate the number and nature of lead and lagged effects of both advertising and promotion in dynamic competitive markets.

Table 4 Change in Relative Allocation Between Advertising and Promotion Due to Marginal Change in Own Interaction Effect

Brands	Tide $\hat{\gamma} \uparrow$	Wisk $\hat{\gamma} \uparrow$	Era $\hat{\gamma} \uparrow$	Solo $\hat{\gamma} \uparrow$	Bold $\hat{\gamma} \uparrow$
Tide's response	Decrease	Decrease	Increase	Decrease	Decrease
Wisk's response	Decrease	Increase	Decrease	Increase	Increase
Era's response	Decrease	Increase	Increase	Increase	Increase
Solo's response	Decrease	Increase	Decrease	Decrease	Increase
Bold's response	Decrease	Increase	Decrease	Increase	Increase

Finally, our model specification misses the support variables such as features and displays, which are shown to interact with price discounts (see van Heerde et al. 2004). While we focused on advertising-promotion interactions in order to develop general methods for estimating dynamic game-theoretic models and computing optimal strategies, we note that no new methodological issues arise in applying the dual methodology to study other interaction effects. Indeed, different model specifications would influence empirical results and yield new marketing insights. Hence, to enrich our understanding of marketing, future researchers should investigate the impact of interaction effects of support activities on optimal budgeting and allocation.

6.2. Dynamic Endogeneity

We conducted the Hausman (1978) test and rejected endogeneity of advertising and promotion. Furthermore, we applied the notions in Engle et al. (1983) and reached the same conclusions (see the unabridged version). Both the approaches reinforce each other to enhance confidence and convergent validity. However, parameter estimates could still be biased due to the possibility of serial correlation causing dynamic endogeneity in advertising and promotion. Thus, future researchers need to develop statistical tests for detecting dynamic endogeneity, characterize the profit impact of such biased estimates, and recommend approaches for de-biasing (see Naik and Tsai 2000 for such efforts in dynamic errors-in-variable problem).

6.3. Role of Retailers

We formulated and examined an extended Lanchester game that describes dynamic competition among multiple manufacturers who incorporate strategic foresight in making intertemporal decisions for multiple marketing activities in the presence of interactions. Our analysis ignores the role of retailers. On the other hand, Moorthy (2005) and Besanko et al. (2005) investigate retailers' pass-through behavior but ignore the effects of intertemporal dynamics, multiple marketing activities, and interaction effects. Future researchers could combine these strands of research by analyzing dynamic competition among manufacturers over time, among retailers, and among manufacturer-retailer dyads. This problem is not only challenging but also an important unsolved problem in marketing science.

7. Concluding Remarks

In dynamic competitive markets, managers should recognize the presence of interactions within marketing activities and between competing brands. To this end, they should incorporate strategic foresight

in their planning by looking forward and reasoning backward in making optimal decisions. By looking forward, each brand manager forecasts his own future plans and anticipates the decisions to be made by other competing brands; by reasoning backward, they deduce their own optimal decisions in response to the best strategies of all other brands. To incorporate the interaction effects and strategic foresight in dynamic oligopoly models, this paper contributes to the marketing literature by creating two methods: marketing-mix algorithm and continuous-discrete estimation method. These dual methodologies enable the computation of optimal marketing-mix plans and the estimation of dynamic game-theoretic models.

We apply the dual methodologies to (a) determine the optimal budget and allocation decisions, and (b) estimate the effectiveness of advertising and promotion and their interaction effect for each of the five brands in a detergents market. We caution the readers that our empirical conclusions about over- and under-spending can change under different, more realistic model specifications. Hence, managers should first fit several alternative specifications using the proposed estimation method (see §6.1) and then determine the specification to retain using appropriate information criteria (Naik and Tsai 2001, 2004). Finally, when managers apply the marketing-mix algorithm to their specific markets, they may discover optimal marketing-mix plans that could appear different from the "business-as-usual" norm. In such situations, managers should use the resulting insights to conduct a small-scale experiment in real markets to generate further market-based evidence and to gain support from other constituencies, both internal (e.g., sales force) and external (e.g., channel members), for successful implementation of marketing-mix strategies.

Acknowledgments

The authors benefited from insightful comments and valuable suggestions of the reviewers, area editor, co-editors of the special issue, the marketing faculty at the Haas School of UC Berkeley, Krannert School of Purdue University, the Graduate School of Management at UC Davis, and the seminar participants at the MSI conference in Boston and the Marketing Science conference in UCLA. They also benefited from the valuable feedback from the participants of the 2nd Annual Control Seminar at UC Davis. Special thanks to professors Sanjay Joshi (Aerospace Engineering), Ahmet Palazoglu (Head of Chemical Engineering), Alan J. Laub (Dean Emeritus of Engineering School), and Zhaojun Bai (Computer Science and Mathematics). Research of the first author was supported in part by the UCD Faculty Research Grant.

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